Hardy spaces in probability and quantum harmonic analysis

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Espaces de Hardy en probabilités et analyse harmonique quantiques

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Résumé

Cette thèse présente quelques résultats de la théorie des probabilités quantiques et de l'analyse harmonique à valeurs operateurs. La thèse est composée des trois parties.

Dans la première partie, on démontre la décomposition atomique des espaces de Hardy de martingales non commutatives. On identifie aussi les interpolés complexes et réels entre les versions conditionnelles des espaces de Hardy et BMO de martingales non commutatives.

La seconde partie est consacrée à l'étude des espaces de Hardy à valeurs opérateurs via la méthode d'ondellettes. Cette approche est similaire à celle du cas des martingales non commutatives. On démontre que ces espaces de Hardy sont équivalents à ceux étudiés par Tao Mei. Par conséquent, on donne une base explicite complétement inconditionnelle pour l'espace de Hardy $H_1(\mathbb{R})$, muni d'une structure d'espace d'opérateurs naturelle.

La troisième partie porte sur l'analyse harmonique sur le tore quantique. On établi les inégalités maximales pour diverses moyennes de sommation des séries de Fourier définies sur le tore quantique et obtient les théorèmes de convergence ponctuelle correspondant. En particulier, on obtient un analogue non commutative du théorème classique de Stein sur les moyennes de Bochner-Riesz. Ensuite, on démontre que les multiplicateurs de Fourier complètement bornés sur le tore quantique coïncident à ceux définis sur le tore classique. Finalement, on présente la théorie des espaces de Hardy et montre que ces espaces possèdent les propriétés des espaces de Hardy usuels. En particulier, on établit la dualité entre H_1 et BMO.

Mots-clefs

Espaces L_p non commutatifs, martingales non commutatives, décomposition atomique, espaces de Hardy et BMO à valeurs opérateurs, ondellettes, tore quantique, series de Fourier, multiplicateurs de Fourier complètement bornés.

Abstract

This thesis presents some results in quantum probability and operator-valued harmonic analysis. The main results obtained in the thesis are contained in the following three parts:

In first part, we prove the atomic decomposition for the Hardy spaces h_1 and $\mathcal{H}_1$ of noncommutative martingales. We also establish that interpolation results on the conditioned Hardy spaces of noncommutative martingales.

The second part is devoted to studying operator-valued Hardy spaces via Meyer's wavelet method. It turns out that this way of approaching these spaces is parallel to that in the noncommutative martingale case. We also show that these Hardy spaces coincide with those introduced and studied by Tao Mei in [52]. As a consequence, we give an explicit completely unconditional bases for Hardy spaces $H_1(\mathbb{R})$ equipped with a natural operator space structure.

The third part deals with with harmonic analysis on quantum tori. We first establish the maximal inequalities for several means of Fourier series defined on quantum tori and obtain the corresponding pointwise convergence theorems. In particular, we prove the noncommutative analogue of the classical Stein theorem on Bochner-Riesz means. Then we prove that L_p completely bounded Fourier multipliers on quantum tori coincide with those on classical tori with equal cb-norms. Finally, we present the H_1-BMO and Littlewood-Paley theories associated with the circular Poisson semigroup over quantum tori.

Keywords

Noncommutative L_p-spaces, noncommutative martingales, atomic decomposition, operator-valued Hardy and BMO spaces, wavelets, quantum tori, Fourier series, completely bounded Fourier multipliers.

Contents

Introduction

L'espace de Hardy est un objet important de l'analyse classique et de la théorie des martingales, et il a beaucoup d'applications à d'autres domaines en mathématique. Si $1 < p < \infty$, on a $L_p = H_p$ avec normes équivalentes par la bornitude de la projection de Riesz. Mais dans le cas $0 < p \leq 1$, la caractérisation des espaces de Hardy est beaucoup plus compliquée. Coifman a d'abord introduit la notion d'atomes [12] dans l'analyse classique. Une question naturelle est comment on peux introduire l'espace de Hardy dans le cadre non commutatif. Notre recherche est basée sur le développment des probabilités quantiques et de l'analyse harmonique non commutative.

L'un des outils principaux dans ces domaines est la théorie des inégalitées de martingales non commutatives. Cette théorie avait déjà été introduite dans les années 70 [17]. Son développement moderne a cependant commencé avec le papier fondateur de Pisier et Xu [69], dans lequel les inégalités de Burkholder-Gundy et le théorème de dualité de Fefferman ont été étendus au cas non commutatif. Depuis, de nombreux résultats classiques ont été transférés avec succès dans le monde non commutatif. Nous renvoyons le lecteur à un livre récent de Xu [93] pour une exposition mise à jour de la théorie des martingales non commutatives.

Parallèlement à la théorie des inégalités non commutatives, l'analyse harmonique non commutative a également fait de grands progrès grâce à des méthodes des espaces d'opérateurs et des inégalités de martingales non commutatives. Nous renvoyons le lecteur notamment au travail de Junge-Le Merdy-Xu [37] sur les sémigroupes de diffusion non commutatifs, aux travaux de Blecher et Labuschagne [5, 6, 7] et de Bekjian-Xu [10] sur les espaces de Hardy non commutatifs définis par des algèbres sous-diagonales, aux travaux de Mei [52] et Chen [11] sur les espaces de Hardy à valeurs opérateurs, aux travaux de Parcet [62] et Mei-Parcet [54] sur la théorie des Caldéron -Zygmund et Littlewood-Paley non commutatives.

Cette thèse est constituée de trois chapitres. Le premier chapitre s'inscrit dans la théorie des martingales non commutatives. On y démontre la décomposition atomique des espaces de Hardy de martingales non commutatives. On identifie aussi les interpolés complexes et réels entre les versions conditionnelles des espaces de Hardy et BMO de martingales non commutatives. Le second chapitre est consacrée à l'étude des espaces de Hardy à valeurs opérateurs via la méthode d'ondellettes. Cette approche est similaire à celle du cas des martingales non commutatives. On démontre que ces espaces de Hardy sont équivalents à ceux étudiés par Tao Mei. Par conséquent, on donne une base explicite complétement inconditionnelle pour l'espace de Hardy $H_1(\mathbb{R})$, muni d'une structure d'espace d'opérateurs naturelle. Le dernier chapitre porte sur l'analyse harmonique sur le tore quantique. On établi les inégalités maximales pour diverses moyennes de sommation des séries de Fourier définies sur le tore quantique et obtient les théorèmes de convergence ponctuelle correspondant. En particulier, on obtient un analogue non commutative du théorème classique de Stein sur les moyennes de Bochner-Riesz. Ensuite, on démontre que les multiplicateurs de Fourier complètement bornés sur le tore quantique coïncident à ceux définis sur le tore classique. Finalement, on présente la théorie des espaces de Hardy et montre que ces espaces possèdent les propriétés des espaces de Hardy usuels. En particulier, on établit la dualité entre H_1 et BMO.

Avant que je présente les résultats principaux. Nous rappelous la définition des espaces L_p non commutatifs. On désigne par $\mathcal{M}$ une algèbre de von Neumann munie d'une trace τ normale, fidèle et semifinie. Soient $S^+_{\mathcal{M}} = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$, où $s(x)$ désigne le support de x. Soit $S_{\mathcal{M}}$ l'espace vectoriel engendré par $S^+_{\mathcal{M}}$. Soient $0 < p < \infty$ et $x \in S_{\mathcal{M}}$. On définit $\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}$. On peut vérifier que $\|\cdot\|_p$ est une (quasi) norme sur $S_{\mathcal{M}}$. L'espace $L_p(\mathcal{M})$ est le complété de $(\mathcal{M}, \|\cdot\|_p)$. Par convention, on définit $L_\infty(\mathcal{M}) = \mathcal{M}$, muni de la norme d'opérateurs.

0.1 Chapitre 1

La décomposition atomique joue un rôle fondamental dans la théorie des martingales classiques et de l'analyse harmonique. Les atomes du cas des martingales sont habituellement définies par des temps d'arrêt. Nous rappelons la définition de ces atomes dans la théorie des martingales classiques. Soient $(\Omega, \mathcal{F}, \mu)$ un espace probabilisé. Soinet $(\mathcal{F}_n)_{n\geq 1}$ une filtration croissante de σ-sous-algèbres de $\mathcal{F}$ telle que $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$. On notera $(\mathcal{E}_n)_{n\geq 1}$ les espérances conditionnelles associées.

On dit qu'une fonction $a \in L_2$ est un atome s'il existes $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ tels que

(i) $\mathcal{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Ces atomes sont appelés atomes simples par Weisz [89], et sont étudiés largement par lui (voir [88] et [89]). On souligne que la décomposition atomique a d'abord été introduite par Coifman [12] en analyse harmonique. C'est Herz [27] qui a introduit la décomposition atomique dans le cas des martingales.

Dans ce chapitre, on va présenter la version non commutative d'atomes et démontrer la décomposition atomique pour les espaces de Hardy de martingales non commutatives. Pour $x \in L_1(\mathcal{M})$, on notera, $r(x)$ et $l(x)$ le support de x à gauche et à droit respectivement. Rappelons que si $x = u|x|$ est la décomposition polaire de x, alors $r(x) = u^*u$ et $l(x) = uu^*$. $r(x)$ (resp. $l(x)$) est aussi la plus petite projection $e \in \mathcal{M}$ telle que $xe = x$ (resp. $ex = x$). Si x est auto-adjoint, alors $r(x) = l(x)$. Soit $x = (x_n)$ une martingale uon commutative relativement à $(\mathcal{M}_n)_{n\geq 1}$. Définissons $dx_n = x_n - x_{n-1}$ pour $n \geq 1$ avec la convention $x_0 = 0$. La suite $dx = (dx_n)$ est appelée la suite des différences de la martingale x. x est une martingale finie s'il existe N tels que $dx_n = 0$ pour tout $n \geq N$. Dans la suite, pour $x \in L_1(\mathcal{M})$, on notera $x_n = \mathcal{E}_n(x)$ pour tout $n \geq 1$.

Nous rappelons la définition des fonctions carrées et des espaces de Hardy pour les martingales non commutatives. Suite à [69], on introduit les versions de ligne et colonne des fonctions carrées d'une martingale finie $x = (x_n)$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

et

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

Soit $1 \le p < \infty$. Définissons $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) comme le complété de l'ensemble des martingales L_p finies pour la norme $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). Les espaces de Hardy de martingales non commutatives sont définis comme suit: Si $1 \le p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}),$$

muni de la norme

$$\|x\|_{\mathcal{H}_p} = \inf\left\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\right\},$$

où l'infimum est pris sur tous les opérateurs $y \in \mathcal{H}_p^c(\mathcal{M})$ et $z \in \mathcal{H}_p^r(\mathcal{M})$ vérifiant $x = y + z$. Pour $2 \le p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}),$$

muni de la norme

$$\|x\|_{\mathcal{H}_p} = \max\left\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\right\}.$$

La raison pour que $\mathcal{H}_p(\mathcal{M})$ soit défini différemment selon $1 \le p < 2$ ou $2 \le p \le \infty$ est présentée dans [69]. Dans cet article-là Pisier et Xu ont montré les inégalités de Burkholder-Gundy non commutatives, qui implique $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ avec normes équivalentes pour tout $1 < p < \infty$.

On considère la version conditionnelle de $\mathcal{H}_p$ introduite dans [43]. Soit $x = (x_n)_{n\ge1}$ une martingale finie dans $L_2(\mathcal{M})$. On pose

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2};$$

et

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}.$$

Ce sont les fonctions carrées conditionnelles de ligne et colonne, respectivement. Soit $0 < p < \infty$. Définit $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) comme le complété de l'ensemble des martingales finies dans $L_\infty(\mathcal{M})$ pour la (quasi) norme $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). Pour $p = \infty$, nous définissons $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) comme l'espace de Banach constitué de martingales $L_\infty(\mathcal{M})$ x telles que $\sum_{k\ge1} \mathcal{E}_{k-1}|dx_k|^2$ (respectivement $\sum_{k\ge1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge pour la topologie d'opérateur faible.

On a besoin aussi de l'espace $\ell_p(L_p(\mathcal{M}))$, l'espace de suites $a = (a_n)_{n\ge1}$ dans $L_p(\mathcal{M})$ telles que

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\ge1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{si } 0 < p < \infty,$$

et

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty < \infty \quad \text{si } p = \infty.$$

Soit $\mathsf{h}_p^d(\mathcal{M})$ le sous espace de $\ell_p(L_p(\mathcal{M}))$ constitué de suites de différences des martingales.

On définit la version conditionnelle des espaces de Hardy comme suit: Si $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}),$$

muni de la (quasi)norme

$$\|x\|_{\mathsf{h}_p} = \inf\left\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r}\right\},$$

où l'infimum est pris sur tous les éléments $w \in \mathsf{h}_p^d(\mathcal{M}), y \in \mathsf{h}_p^c(\mathcal{M})$ et $z \in \mathsf{h}_p^r(\mathcal{M})$ vérifiant $x = w + y + z$. Pour $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}),$$

muni de la norme

$$\|x\|_{\mathsf{h}_p} = \max\left\{\|x\|_{\mathsf{h}_p^d}, \|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}\right\}.$$

Les inégalités de Burkholder non commutatives démontrées dans [43] affirment que

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \tag{0.1.1}$$

avec normes équivalentes pour tout $1 < p < \infty$.

Puisqu'il y a deux fonctions carrées, il existe deux types d'atomes dans le cas non commutatif.

Définition 0.1.1. *On dit que $a \in L_2(\mathcal{M})$ est un $(1,2)_c$-atome associé à $(\mathcal{M}_n)_{n\geq 1}$, s'il existe $n \geq 1$ et une projection $e \in \mathcal{M}_n$ tels que*

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{-1/2}$.

En remplaçant (ii) *par* (ii)′ $l(a) \leq e$, *on obtient la notion d'un* $(1,2)_r$*-atome.*

Ici, les $(1,2)_c$-atomes et $(1,2)_r$-atomes sont les analogues non commutatifs des $(1,2)$-atomes de martingales classiques, et sont démontrés d'être adaptés aux espaces de Hardy de colonne et ligne. De l'autre côté, à cause de la non-commutativité, certaines constructions basées sur les temps d'arrêt dans le cas classique ne sont pas valables dans le cadre non commutatif, notre approche à la décomposition atomique pour les espaces de Hardy conditionnelles de martingales non commutatives passe par la dualité $\mathsf{h}_1 - \mathsf{bmo}$. Rappelons que l'égalité de dualité $(\mathsf{h}_1)^* = \mathsf{bmo}$ a été établie indépendamment dans [34] et [61]. Notre approche n'est malheureusement pas constructive. En résumé, on a le théorème suivant:

Théorème 0.1.2. *On a*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) \quad \textit{avec normes équivalentes.}$$

Plus précisement, si $x \in \mathsf{h}_1(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}_1^{\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathsf{h}_1^{\mathrm{at}}}.$$

Remarque 0.1.3. *Dans un travail récent [30], G. Hong and T. Mei étendent ce résultat et établissent la décomposition q-atomique, pour tout $1 < q \leq \infty$, en utilisaut leurs inégalité de John-Nirenberg pour les martingales non commutatives.*

L'autre résultat principal de ce chapitre concerne l'interpolation des espaces de Hardy conditionnelles h_p. L'idée principale de notre preuve est inspirée par une norme équivalente de $\mathsf{h}_p, 0 < p \leq 2$ introduite par Herz [28] dans le cas commutatif. On traduit cette quasi norme au cadre non commutatif afin d'obtenir une nouvelle caractérisation de h_p, $0 < p \leq 2$, qui est plus pratique pour l'interpolation. On a le théorème d'interpolation suivant:

Théorème 0.1.4. *Soit $1 < p < \infty$. Alors*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p(\mathcal{M}) \quad \textit{avec normes équivalentes.}$$

0.2 Chapitre 2

Dans ce chapitre, on exploite les ondelettes de Meyer à l'étude des espaces de Hardy à valeurs opérateurs. Une base d'ondelettes de $L_2(\mathbb{R})$ est un système orthonormal complet $(w_I)_{I\in\mathcal{D}}$, où $\mathcal{D}$ désigne l'ensemble des intervalles dyadiques dans $\mathbb{R}$, w est une fonction de Schwartz vérifiant les proprietés nécessaires dans la construction de Meryer [49], et

$$w_I(x) \doteq \frac{1}{|I|^{\frac{1}{2}}} w(\frac{x-c_I}{|I|}),$$

où c_I est le centre de I. Les faits importants dont on aura besoin sur la base d'ondelettes sont l'orthogonalité entre w_I différente, $\|w\|_{L_2(\mathbb{R})} = 1$ et la régularité de w,

$$\max(|w(x)|, |w'(x)|) \lesssim (1+|x|)^{-m}, \quad \forall m \geq 2.$$

L'analogie entre ondelettes et martingales dyadiques est bien connue. L'observation clef est le parallélisme suivant:

$$\sum_{|I|=2^{-n+1}} \langle f, w_I\rangle w_I \sim df_n,$$

où df_n désigne la n-ème différence de la martingale dyadique f. A l'aide de cette relation et l'orthogonalité de $(w_I)_{I\in\mathcal{D}}$, on peut utiliser la méthode de martingales non commutatives pour étudier l'analyse harmonique à valeurs opérateurs. Nous remarquons que Mei dans [52] a établi la thèorie des espaces de Hardy à valeur opérateurs par la méthode de la théorie de Littlewood-Paley; mais notre approche semblerait plus simple que celle de Mei.

Dans ce chapitre, pour simplifier les notations, on désigne $\mathcal{N} = L_\infty(\mathbb{R})\bar{\otimes}\mathcal{M}$. Comme dans le cas classique, pour $f \in S_{\mathcal{N}}$, on définit les deux fonctions carrées de Littlewood-Paley comme suit

$$S_c(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{0.2.1}$$

$$S_r(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f^*, w_I\rangle|^2}{|I|} \mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{0.2.2}$$

Pour $1 \leq p < \infty$, on considère

$$\|f\|_{\mathcal{H}_p^c} = \|S_c(f)\|_{L_p(\mathcal{N})},$$

$$\|f\|_{\mathcal{H}_p^r} = \|S_r(f)\|_{L_p(\mathcal{N})}.$$

Ce sont les normes. Donc on définit l'espaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{R},\mathcal{M})$) comme le complété de $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$). Nous définissons maintenant l'espaces de Hardy à valeurs opérateurs comme suit: pour $1 \leq p < 2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) + \mathcal{H}_p^r(\mathbb{R},\mathcal{M}), \tag{0.2.3}$$

muni de la norme

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}_p^c} + \|h\|_{\mathcal{H}_p^r} : f = g+h, g \in \mathcal{H}_p^c, h \in \mathcal{H}_p^r\}$$

et pour $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R},\mathcal{M}), \tag{0.2.4}$$

muni de la norme

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\}.$$

Pour $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, on pose

$$\|\varphi\|_{\mathcal{BMO}^c} = \sup_{J\in\mathcal{D}} \Big\|\big(\frac{1}{|J|}\sum_{I\subset J} |\langle\varphi, w_I\rangle|^2\big)^{\frac12}\Big\|_{\mathcal{M}} \tag{0.2.5}$$

et

$$\|\varphi\|_{\mathcal{BMO}^r} = \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

Définissons

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\}$$

et

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^r} < \infty\}.$$

Ce sont des espaces de Banach modulo les fonctions constantes. On définit alors

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

Comme dans le cas de martingales [43], on peut aussi définir $L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ pour tout $2 < p \leq \infty$. Pour $\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, pose

$$\|\varphi\|_{L_p^c\mathcal{MO}} = \Big\|\big(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle\varphi, w_I\rangle|^2\big)_k\Big\|^{\frac12}_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} \tag{0.2.6}$$

et

$$\|\varphi\|_{L_p^r\mathcal{MO}} = \|\varphi^*\|_{L_p^c\mathcal{MO}},$$

où I_k^x est l'interval dyadique unique avec la longue 2^{-k+1} qui contient x. On utilisera la convention adoptée dans [45] pour la norme de $L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)$. Ainsi

$$\Big\|\big(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle\varphi, w_I\rangle|^2\big)_k\Big\|_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} = \Big\|{\sup_k}^+ \frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle\varphi, w_I\rangle|^2\Big\|_{L_{\frac{p}{2}}(\mathcal{N})}.$$

On féfinit

$$L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^c\mathcal{MO}} < \infty\}$$

et

$$L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_r^c\mathcal{MO}} < \infty\}.$$

Définissons

$$L_p\mathcal{MO}(\mathbb{R},\mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \cap L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}).$$

Remarquons que $L_\infty^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$.

Alors on a les thèorèmes de dualité et d'interpolation suivants:

Théorème 0.2.1. *On a*

$$(\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \tag{0.2.7}$$

avec normes équivalentes.

De même, cette dualité est encore vraie entre $\mathcal{H}_1^r$ et $\mathcal{BMO}^r$, entre $\mathcal{H}_1$ et $\mathcal{BMO}$ avec normes équivalentes.

Théorème 0.2.2. *Soit* $1 < p < 2$. *On a*

$$(\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^* = L_{p'}^c \mathcal{MO}(\mathbb{R}, \mathcal{M}) \tag{0.2.8}$$

avec normes équivalentes.

De même, cette dualité est encore vraie entre $\mathcal{H}_p^r$ *et* $L_{p'}^r$, *entre* $\mathcal{H}_p$ *et* $L_{p'}\mathcal{MO}$ *avec normes équivalentes.*

Théorème 0.2.3. *Pour tout* $1 < p < \infty$, *on a*

$$(\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{R}, \mathcal{M}). \tag{0.2.9}$$

Théorème 0.2.4. *Les résultats suivants sont vrais avec normes équivalent:*

i *Soit* $1 \leq q < p < \infty$, *on a*

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} = L_p(\mathcal{N}). \tag{0.2.10}$$

ii *Soit* $1 < q < p \leq \infty$, *on a*

$$[\mathcal{H}_1(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p'}{q'}} = L_q(\mathcal{N}). \tag{0.2.11}$$

iii *Soit* $1 < p < \infty$, *on a*

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{p}} = L_p(\mathcal{N}). \tag{0.2.12}$$

Soient $H_p^c(\mathbb{R}, \mathcal{M})$ et $BMO^c(\mathbb{R}, \mathcal{M})$ les espace de Hardy et BMO de [52]. On a le résultat suivant.

Théorème 0.2.5. *On a*

$$\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) = BMO^c(\mathbb{R}, \mathcal{M})$$

avec normes équivalents. Le résultat similaire est aussi vrai pour l'espace de ligne.Par conséquent, $\mathcal{BMO}(\mathbb{R}, \mathcal{M}) = BMO(\mathbb{R}, \mathcal{M})$ *avec norems équivalent.*

0.3 Chapitre 3

Soit $d \geq 2$ et $\theta = (\theta_{kj})$ une matrice $d \times d$ skew-symétrique réelle. Rappelons que le tore non commutatif de dimension d est la C^*-algèbre universelle, generée par d vecteurs unitaires $U_1, \ldots, U_d$ vérifiant la relation de commutation suivante

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \; j, k = 1, \ldots, d. \tag{0.3.1}$$

Il exist une trace fidèle τ sur $\mathcal{A}_\theta$. Soit $\mathbb{T}_\theta^d$ l'algèbre de von Neumann obtenu par la représentation GNS de τ. On dit que $\mathbb{T}_\theta^d$ est le d-tore quantique associé au θ. Remarquons que si $\theta = 0$, alors $\mathcal{A}_\theta = C(\mathbb{T}^d)$ et $\mathbb{T}_\theta^d = L_\infty(\mathbb{T}^d)$, où $\mathbb{T}^d$ est le d-tore habituel. En conséquce, un d-tore quantique est une déformation de d-tore habituel, C'est donc nature d'espère que $\mathbb{T}_\theta^d$ partage beaucoup de propriétés avec $\mathbb{T}^d$. Ce la se passe en effet dans le cas de la géométrie différentielle, comme montrés dans les travaux de Connes et ses collaborateurs. Cependant, peu est fait en ce qui concerne l'analyse. A notre connaissance, jusqu'à maintenant, seuls le théorème de convergence de moyenne de séries de Fourier quantiques par la sommation de Fejèr cubique a été démontré au niveau de C^*-algèbre (cf. [86, 87]), et de l'autre côté, l'analogue du tore quantique des inégalités de Sobolev n'a été obtenu que dans le cas d'espace de Hilbert (cf. [80]). La raison peut être expliqué par de nombreues difficultés rencontrées en traitant les espaces L_p non commutatifs. Par exemple, le moyen habituel de montrer les théorèmes de convergence ponctuelle est d'établir les inegalités maximales associées. Mais l'étude des inégalités maximales est une des parties les plus subtiles et difficiles dans l'analyse non commutative.

Ce chapitre est le premier d'un poject qui a pour but de développer analyse sur le tore quantique et plus généralement sur les produits croisés to dus des groupes moyennables. Notre but ici est d'étudier quelques aspects importants de l'annalyse harmonique sur $\mathbb{T}_\theta^d$. Les sujets auxquels nous nous sommes intéressés sont suivants:

i) *Convergence de séries de Fourier.* On établit les inégalités maximales pour plusieurs moyennes de sommation de séries de Fourier définies sur le tore quantique et obtient aussi les théorèmes de convergence ponctuelle correspondan. En particulier, on démontre un analogue non commutatif du théorème classique de Stein sur les moyennes de Bochner-Riesz.

ii) *Multipliacteurs de Fourier.* On prouve que les multiplicateurs de Fourier complétements bornés sur le tore quantique sont exactement ceux sur le tore classique avec cb-normes équales.

iii) *Espaces de Hardy et* BMO. On présente la dualité entre H_1 et BMO, et la théorie de Littlewood-Paley associés au semigrupe de Poisson circulaire sur le tore quantique.

Notre stratégie pour résoudre ces problèmes est de les transférer aux analogues dans le cas à valeurs opérateurs sur le tore classique. Pour expliquer nos résultats principaux, on a besoin de quelques notations. Soient $1 \leq p \leq \infty$ et $x \in L_p(\mathbb{T}_\theta^d)$. x admet une série formelle de Fourier:

$$x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m, \tag{0.3.2}$$

où

$$\hat{x}(m) = \tau(x(U^m)^*),\ m \in \mathbb{Z}^d, \tag{0.3.3}$$

est appelé le $m^{\text{ère}}$ coefficient de Fourier de x. L'un des sujets principaux de l'analyse harmonique est d'étudier dans quel sens la série sur le côté droit de (0.3.1) converges vers x. Comme dans le cas classique, on considèreles trois types de moyennes de sommation:

1) La *moyenne de Cesàro cubique*

$$F_N[x] = \sum_{\substack{m \in \mathbb{Z}^d, \\ |m|_\infty \leq N}} \Big(1 - \frac{|m_1|}{N+1}\Big) \cdots \Big(1 - \frac{|m_d|}{N+1}\Big) \hat{x}(m) U^m, \quad N \geq 0. \tag{0.3.4}$$

2) La *moyenne de Poisson cubique*

$$P_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_1} U^m, \quad 0 \leq r < 1. \tag{0.3.5}$$

3) La *moyenne de Poisson circulaire*

$$\mathbb{P}_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m, \quad 0 \leq r < 1. \tag{0.3.6}$$

4) Soit Φ une fonction continue sur $\mathbb{R}^d$ avec $\Phi(0) = 1$. Définition

$$\Phi^\varepsilon[x] = \sum_{m\in\mathbb{Z}^d} \Phi(\varepsilon m)\hat{x}(m)U^m, \quad \varepsilon > 0.$$

On va toujour imposer la condition suivante sur Φ:

$$\begin{cases} \Phi(s) = \hat{\varphi}(s) \quad \text{avec} \displaystyle\int_{\mathbb{R}^d} \varphi(s)ds = 1; \\ |\Phi(s)| + |\varphi(s)| \le A(1+|s|)^{-d-\delta}, \ \forall s \in \mathbb{R}^d, \end{cases} \tag{0.3.7}$$

pour certaines $A, \delta > 0$ (cf. [83, p. 253]).

Ici, $|m|_p = (\sum_{j=1}^d |m_j|^p)^{1/p}$ pour $1 \le p < \infty$, et $|m|_\infty = \sup_{1\le j\le d} |m_j|$.

Ce qui suit est un de nos rèsultats principaux:

Théorème 0.3.1. (1) *Soit $x \in L_1(\mathbb{T}_\theta^d)$. Alors pour tout $\alpha > 0$, il existe une projection $e \in \mathbb{T}_\theta^d$ telle que*

$$\sup_{N\ge 0} \left\| eF_N[x]e \right\|_\infty \le \alpha \quad et \quad \tau(e^\perp) \le C_d \frac{\|x\|_1}{\alpha}.$$

(2) *Soit $1 < p \le \infty$. Alors*

$$\left\| {\sup_{N\ge 0}}^+ F_N[x] \right\|_p \le C_d \frac{p^2}{(p-1)^2} \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}_\theta^d).$$

Les deux assertions sont encore vraie pour les trois autre moyens de sommation P_r, $\mathbb{P}_r$ et Φ^ε. Dans le cas Φ^ε, la constante C_d dépend aussi des deux constantes dans (3.2.1).

Comme d'habitude, les inégalités maximales dans Théorème 0.3.1 devrait impliquer les théorèmes de convergence ponctuelle. En adaptant les arguments de M. Junge et Q. Xu [45], on obtient en effet le rèsultat suivant:

Théorème 0.3.2. *Soit $1 \le p \le \infty$ et $x \in L_p(\mathbb{T}_\theta^d)$. Alors $F_N[x] \xrightarrow{b.a.u.} x$ lorsque $N \to \infty$. Pour $2 \le p \le \infty$ la b.a.u convergence précédente peut être renforcés par la a.u convergence. Le résultat similaire est encore vrai si on remplace F_N par les trois autres moyennes de sommation P_r, $\mathbb{P}_r$ et Φ^ε.*

Ici $x_n \xrightarrow{b.a.u} x$ signifie que (x_n) converge à x bilatéralement presque uniformément (b.a.u, pour abbrévation) et $x_n \xrightarrow{a.u} x$ signifie presque uniformément. Ces notions ont été introduites par Lance [48] (voir aussi [45] pour les détails).

On peut aussi considérer la *moyenne de Cesàro circulaire*:

$$\mathbb{F}_N[x] = \sum_{\substack{m\in\mathbb{Z}^d,\\ |m|_2\le N}} \Big(1 - \frac{|m|_2}{N+1}\Big)\hat{x}(m)U^m, \quad N \ge 0. \tag{0.3.8}$$

Cependant, les résultats précédents ne sont plus vrais pour $\mathbb{F}_N$ même dans le cas classique [83]. En effet, on a besoin de considérer, à la place, la *moyenne de Bochner-Riesz* d'ordre α :

$$B_R^\alpha[x] = \sum_{|m|_2\le R} \Big(1 - \frac{|m|_2^2}{R^2}\Big)^\alpha \hat{x}(m)U^m. \tag{0.3.9}$$

Si $\alpha > (d-1)/2$, le noyau de la moyenne de Bochner-Riesz dans ce cas-là est une identité d'approximation, on obtient alors les résultats de la convergence de moyenne, des inégalités maximales et de la convergence ponctuelles. En fait, les (3) dans les Théorèmes 0.3.1 et 0.3.2, et le résultat similaire pour la *moyenne de Bochner-Riesz* avec l'ordre α plus grande que $(d-1)/2$, seront démontrés comme une conséquence du cas général concernant une identité d'approximation déterminée par une fonction continue sur $\mathbb{R}^d$ avec une condition supplémentaire sur le comportement asymptotique à l'infini.

Pour le cas $\alpha \leq (d-1)/2$, on a le théorème suivant, qui est la généralisation du théorème de Stein [81] dans le cadre du tore quantique.

Théorème 0.3.3. *Soit $1 < p < \infty$ et $\alpha > (d-1)|\frac{1}{2} - \frac{1}{p}|$. Alors*

(1) *Pour tout $x \in L_p(\mathbb{T}_\theta^d)$,*

$$\left\|{\sup_{R>0}}^{+} B_R^\alpha[x]\right\|_p \leq C_p \|x\|_p.$$

(2) $\lim_{R\to\infty} B_R^\alpha[x] = x$ *dans* $L_p(\mathbb{T}_\theta^d)$.

(3) *Pour tout $x \in L_p(\mathbb{T}_\theta^d)$, $B_R^\alpha[x] \xrightarrow{b.a.u} x$ si $R \to \infty$.*

Nous nous tournons vers le second thème de ce chapitre. On discute de multiplicateurs de Fourier complètement bornés sur le tore quantique. Notre motivation vient de la théorie des multiplicateurs de Fourier sur le tore classique [20, 83].

On définit les multiplicateur de Fourier sur le tore quantique naturellement. Soit $\phi = (\phi_m)_{m\in\mathbb{Z}^d}$. On définit T_ϕ par

$$\widehat{T_\phi x}(m) = \phi_m \hat{x}(m), \quad \forall m \in \mathbb{Z}^d,$$

pour tout $x \in \mathbb{T}_\theta^d$. On dit que ϕ est un multiplicateur bornée dans L_p (resp. c.b. L_p multiplicateur) sur le tore quantique $\mathbb{T}_\theta^d$, si l'opérateur T_ϕ extend à une application bornée (resp. c.b.) dans $L_p(\mathbb{T}_\theta^d)$. Soient $\mathrm{M}(L_p(\mathbb{T}_\theta^d))$ et $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$ l'ensemble des multiplicateurs bornés et complètement bornés dans L_p, respectivement. On a le théorème suivant:

Théorème 0.3.4. *Soit $1 < p \leq \infty$. Alors $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = \mathrm{M}_{\mathrm{cb}}(L_p(L_\infty(\mathbb{T}^d)))$ avec* cb-*normes équales.*

Le troisième thème de ce chapitre traite la dualité entre H^1 et BMO, et la théorie de Littlewood-Paley associés au semigroupe de Poisson circulaire $\mathbb{P}_r$ sur le tore quantique. Pour tout $x \in \mathbb{T}_\theta^d$, nous définissons

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr\Big)^{1/2}.$$

Pour $1 \leq p < \infty$, on pose

$$\|x\|_{\mathrm{H}_p^c} = |\hat{x}(\mathbf{0})| + \|G_c(x)\|_{L_p(\mathbb{T}_\theta^d)}.$$

C'est une norme sur T_θ^d (cf. e.g. [37]). On définit l'espace de Hardy colonne $\mathrm{H}_p^c(\mathbb{T}_\theta^d)$ comme le complété de $\mathbb{T}_\theta^d$ relativement à cette norme. L'espace de Hardy ligne $\mathrm{H}_p^r(\mathbb{T}_\theta^d)$ est défini comme l'espace de x tel que $x^* \in \mathrm{H}_p^c(\mathbb{T}_\theta^d)$ muni de la norme naturelle. Les espaces de Hardy sont définis comme suit: Si $1 \leq p < 2$,

$$\mathrm{H}_p(\mathbb{T}_\theta^d) = \mathrm{H}_p^c(\mathbb{T}_\theta^d) + \mathrm{H}_p^r(\mathbb{T}_\theta^d)$$

muni de la norme de somme

$$\|x\|_{\mathrm{H}_p} = \inf\{\|G_c(a)\|_p + \|G_r(b)\| \,:\, x = a + b, a \in \mathrm{H}_p^c(\mathbb{T}_\theta^d), b \in \mathrm{H}_p^r(\mathbb{T}_\theta^d)\},$$

et si $2 \le p < \infty$,

$$\mathrm{H}_p(\mathbb{T}_\theta^d) = \mathrm{H}_p^c(\mathbb{T}_\theta^d) \cap \mathrm{H}_p^r(\mathbb{T}_\theta^d)$$

muni de la norme d'intersection

$$\|x\|_{\mathrm{H}_p} = \max\big\{\|G_c(x)\|_p,\ \|G_r(x)\|_p\big\}.$$

On va aussi travaille avec les espaces de BMO sur $\mathbb{T}_\theta^d$. On pose

$$\mathrm{BMO}^c(\mathbb{T}_\theta^d) = \{x \in L_2(\mathbb{T}_\theta^d) \,:\, \sup_r \big\|\mathbb{P}_r\big[|x - \mathbb{P}_r[x]|^2\big]\big\|_\infty < \infty\}$$

muni de la norme

$$\|x\|_{\mathrm{BMO}^c} = \max\big\{|\hat{x}(\mathbf{0})|,\ \sup_r \big\|\mathbb{P}_r\big[|x - \mathbb{P}_r[x]|^2\big]\big\|_\infty^{1/2}\big\}.$$

$\mathrm{BMO}^r(\mathbb{T}_\theta^d)$ est défini comme l'espace de x tel que $x^* \in \mathrm{BMO}^c(\mathbb{T}_\theta^d)$ avec la norme $\|x\|_{\mathrm{BMO}^r} = \|x^*\|_{\mathrm{BMO}^c}$. The mixture $\mathrm{BMO}(\mathbb{T}_\theta^d)$ est l'intersection de ces deux espaces:

$$\mathrm{BMO}(\mathbb{T}_\theta^d) = \mathrm{BMO}^c(\mathbb{T}_\theta^d) \cap \mathrm{BMO}^r(\mathbb{T}_\theta^d)$$

avec la norme d'intersection.

Les définitions ci-dessus sont motivées par les espaces de Hardy de martingales non commutative ([43, 69]) et de semigroupes Markovien quantique ([37, 38, 52]). Les résultats principe de cette partie sont résumés dans le théorème suivant, qui démontre que les espaces de Hardy sur $\mathbb{T}_\theta^d$ possèdent les propriétés des espaces de Hardy habituel, comme espéré.

Theorem 0.3.1. i) *Soit* $1 < p < \infty$. *Alors* $\mathrm{H}_p(\mathbb{T}_\theta^d) = L_p(\mathbb{T}_\theta^d)$ *avec normes équivalentes.*

ii) *L'espace dual de* $\mathrm{H}_1^c(\mathbb{T}_\theta^d)$ *est* $\mathrm{BMO}^c(\mathbb{T}_\theta^d)$ *avec normes équivalentes par la parenthèse de dualité*

$$\langle x,\ y\rangle = \tau(xy^*), \quad x \in L_2(\mathbb{T}_\theta^d),\ y \in \mathrm{BMO}^c(\mathbb{T}_\theta^d).$$

La même assertion est encore vraie pour les deux types d'espaces.

iii) *Soit* $1 < p < \infty$. *Alors*

$$(\mathrm{BMO}^c(\mathbb{T}_\theta^d),\ \mathrm{H}_1^c(\mathbb{T}_\theta^d))_{1/p} = \mathrm{H}_p^c(\mathbb{T}_\theta^d) = (\mathrm{BMO}^c(\mathbb{T}_\theta^d),\ \mathrm{H}_1^c(\mathbb{T}_\theta^d))_{1/p,p}$$

avec normes équivalentes, où $(\,\cdot\,,\ \cdot\,)_{1/p}$ *et* $(\,\cdot\,,\ \cdot\,)_{1/p,p}$ *dénote respectivement les foncteurs d'interpolation complexe et réelle.*

iv) *Soit* $1 < p < \infty$ *et* $X_0 \in \{\mathrm{BMO}(\mathbb{T}_\theta^d),\ L_\infty(\mathbb{T}_\theta^d)\}$, $X_1 \in \{\mathrm{H}_1(\mathbb{T}_\theta^d),\ L_1(\mathbb{T}_\theta^d)\}$. *Alors*

$$(X_0,\ X_1)_{1/p} = L_p(\mathbb{T}_\theta^d) = (X_0,\ X_1)_{1/p,p}$$

avec normes équivalentes.

Introduction

Hardy space is an important concept of classical analysis and martingale theory, and it has many applications to other mathematic field. When $1 < p < \infty$, by the boundness of Riesz projection, we have $L_p = H_p$ isometrically. But in the case of $0 < p \leq 1$, the characterization of H_p spaces is much more complicated. To this end, Coifman first introduced the concept of atoms [12] in the classical analysis. Parallel to this, Herz [27] proved the atom decomposition of Hardy space of martingale. A natural question is how we can define Hardy spaces in various noncommutative setting.

This theory had been already initiated in the 1970's [17]. Its modern period of development has begun with Pisier and Xu's seminal paper [69] in which the authors established the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem between H_1 and BMO. Since then many classical results have been successfully transferred to the noncommutative world (see [43], [46], [52]). We refer to a recent book by Xu [93] for an up-to-date exposition of theory of noncommutative martingales.

With parallel to the theory of noncommutative inequalities, noncommutative harmonic analysis has also made great advances by using the method of operator space and noncommutative martingale inequality. We refer the reader notably to the recent works by Junge-Le Merdy-Xu [37] on noncommutative diffusion semigroups, by Blecher and Labuschagne [5, 6, 7] and Bekjan-Xu [10] on noncommutative Hardy spaces, by Mei [52] and Chen [11] on operator-valued Hardy spaces, and by Parcet [62] and Mei-Parcet [54] on noncommutative Caldrón-Zygmund and Littlewood-Paley theories.

This thesis consists of three chapters, which is based on the recent development of quantum probability and noncommutative harmonic analysis. The first chapter presents a joint work with T.Bekjan, Z.Chen, M.Perrin, entitled "Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales", which can be viewed as a part of noncommutative martingale theory. The content of the second chapter is devoted to the study on theory of vector-valued Hardy spaces. This chapter is a joint work with G.Hong entitled "Wavelet approach to operator-valued Hardy spaces". The last chapter is concerned with harmonic analysis on the quantum tori, which is a joint work with Z.Chen and Q.Xu entitled "Harmonic analysis on quantum tori".

Before we give the main results, we recall the definition of L_p spaces. $\mathcal{M}$ will always denote a von Neumann algebra with a normal faithful normalized trace τ. Let $S^+_{\mathcal{M}} = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$, and $S_{\mathcal{M}}$ is the linear expansion of $S^+_{\mathcal{M}}$. Let $0 < p < \infty$ and $x \in S$. Define $\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}$. We can prove that if $p \geq 1$, $\|\cdot\|_p$ is a norm, while $p < 1$, it is a p norm.

0.1 Chapter 1

Atomic decomposition plays a fundamental role in the classical martingale theory and harmonic analysis. Atoms for martingales are usually defined in terms of stopping times. Let us recall this in classical martingale theory. Given a probability space $(\Omega, \mathcal{F}, \mu)$, let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing filtration of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\cup_n \mathcal{F}_n\big)$ and let $(\mathcal{E}_n)_{n\geq 1}$ denote the corresponding family of conditional expectations. An $\mathcal{F}$-measurable function $a \in L_2$ is said to be an *atom* if there exist $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Such atoms are called *simple atoms* by Weisz [89] and are extensively studied by him (see [88] and [89]). Let us point out that atomic decomposition was first introduced in harmonic analysis by Coifman [12]. It is Herz [27] who initiated atomic decomposition for martingale theory.

In this chapter, we will present the noncommutative version of atoms and prove that atomic decomposition for the Hardy spaces of noncommutative martingales is valid for these atoms. For $x \in L_p(\mathcal{M})$ we denote by $r(x)$ and $l(x)$ the right and left supports of x, respectively. Recall that if $x = u|x|$ is the polar decomposition of x, then $r(x) = u^*u$ and $l(x) = uu^*$. $r(x)$ (resp. $l(x)$) is also the least projection e such that $xe = x$ (resp. $ex = x$). If x is selfadjoint, $r(x) = l(x)$. Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $dx = (dx_n)$ is called the *martingale difference sequence* of x. x is called a *finite martingale* if there exists N such that $dx_n = 0$ for all $n \geq N$. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Let us now recall the definitions of the square functions and Hardy spaces for noncommutative martingales. Following [69], we introduce the column and row versions of square functions relative to a (finite) martingale $x = (x_n)$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

Let $1 \leq p < \infty$. Define $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) as the completion of all finite L_p-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). The Hardy space of noncommutative martingales is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf\big\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\big\},$$

where the infimum is taken over all $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$ such that $x = y + z$. For $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

The reason that $\mathcal{H}_p(\mathcal{M})$ is defined differently according to $1 \le p < 2$ or $2 \le p \le \infty$ is presented in [69]. In that paper Pisier and Xu prove the noncommutative Burkholder-Gundy inequalities which imply that $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ with equivalent norms for $1 < p < \infty$.

We now consider the conditioned version of $\mathcal{H}_p$ developed in [43]. Let $x = (x_n)_{n\ge1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2};$$

and

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}.$$

These will be called the column and row conditioned square functions, respectively. Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi)norm $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). For $p = \infty$, we define $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) as the Banach space of the $L_\infty(\mathcal{M})$-martingales x such that $\sum_{k\ge1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\ge1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology.

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\ge1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\ge1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{if } 0 < p < \infty,$$

and

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty \quad \text{if } p = \infty.$$

Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.

We define the conditioned version of martingale Hardy spaces as follows: If $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M})$$

equipped with the (quasi)norm

$$\|x\|_{\mathsf{h}_p} = \inf\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r}\},$$

where the infimum is taken over all $w \in \mathsf{h}_p^d(\mathcal{M}), y \in \mathsf{h}_p^c(\mathcal{M})$ and $z \in \mathsf{h}_p^r(\mathcal{M})$ such that $x = w + y + z$. For $2 \le p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p} = \max\{\|x\|_{\mathsf{h}_p^d}, \|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}\}.$$

The noncommutative Burkholder inequalities proved in [43] state that

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \tag{0.1.1}$$

with equivalent norms for all $1 < p < \infty$.

Due to the fact that there are two kinds of Hardy spaces, there correspond two kinds atoms.

Definition 0.1.1. *$a \in L_2(\mathcal{M})$ is said to be a $(1,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that*

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{-1/2}$.

Replacing (ii) *by* (ii)$'$ $l(a) \leq e$, *we get the notion of a $(1,2)_r$-atom.*

Here, $(1,2)_c$-atoms and $(1,2)_r$-atoms are noncommutative analogues of $(1,2)$-atoms for classical martingales, which are proved to be suitable for the column (resp. row) Hardy spaces. On the other hand, due to the noncommutativity some basic constructions based on stopping times for classical martingales are not valid in the noncommutative setting, our approach to the atomic decomposition for the conditioned Hardy spaces of noncommutative martingales is via the $\mathsf{h}_1 - \mathsf{bmo}$ duality. Recall that the duality equality $(\mathsf{h}_1)^* = \mathsf{bmo}$ was established independently by [34] and [61]. However, this method does not give an explicit atomic decomposition. In summery, we get following theorem:

Theorem 0.1.2. *We have*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) \quad \text{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}_1(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}_1^{\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathsf{h}_1^{\mathrm{at}}}.$$

Remark 0.1.3. A recent work of G.Hong and T.Mei [30] extend this 2-atom decomposition to the q-atom decomposition, $1 < q \leq \infty$, by using the John-Nirenberg inequality for noncommutative martingale.

The other main result of this chapter concerns the interpolation of the conditioned Hardy spaces h_p. Such kind of interpolation results involving Hardy spaces of noncommutative martingales first appear in Musat's paper [50] for the spaces $\mathcal{H}_p$. We will present an extension of these results to the conditioned case.The main idea is inspired by an equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$ introduced by Herz [28] in the commutative case. We translate this quasinorm to the noncommutative setting to obtain a new characterization of h_p, $0 < p \leq 2$, which is more convenient for interpolation. By this way we show following interpolation theorem:

Theorem 0.1.4. *Let $1 < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p(\mathcal{M}).$$

0.2 Chapter 2

In this chapter, we exploit Meyer's wavelet methods to the study of the operator-valued Hardy spaces. A wavelet basis of $L_2(\mathbb{R})$ is a complete orthonormal system $(w_I)_{I\in\mathcal{D}}$, where $\mathcal{D}$ denotes the collection of all dyadic intervals in $\mathbb{R}$, w is a Schwartz function satisfying the properties needed for Meryer's construction in [49], and

$$w_I(x) \doteq \frac{1}{|I|^{\frac{1}{2}}} w(\frac{x - c_I}{|I|}),$$

where c_I is the center of I. The central facts that we will need about the wavelet basis are the orthogonality between different w_I's, $\|w\|_{L_2(\mathbb{R})}=1$ and the regularity of w,

$$\max(|w(x)|,|w'(x)|)\lesssim(1+|x|)^{-m},\quad \forall m\geq 2.$$

The analogy between wavelets and dyadic martingales is well known. The key observation is the following parallelism:

$$\sum_{|I|=2^{-n+1}}\langle f,w_I\rangle w_I\sim df_n,$$

where df_n denotes n-th dyadic martingale difference of f. With this relationship and the orthogonality of the $(w_I)_{I\in\mathcal{D}}$, we can use the method of noncommutative martingale to the operator-valued harmonic analysis. Note that Mei has established the operator-valued Hardy spaces theory [52], but our approach is much simpler than his.

In this chapter, for simplicity, we denote $L_\infty(\mathbb{R})\bar{\otimes}\mathcal{M}$ by $\mathcal{N}$. As in the classical case, for $f\in S_{\mathcal{N}}$, we define the two Littlewood-Paley square functions as

$$S_c(f)(x)=\Big(\sum_{I\in\mathcal{D}}\frac{|\langle f,w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac12}. \tag{0.2.1}$$

$$S_r(f)(x)=\Big(\sum_{I\in\mathcal{D}}\frac{|\langle f^*,w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac12}. \tag{0.2.2}$$

For $1\leq p<\infty$, define

$$\|f\|_{\mathcal{H}_p^c}=\|S_c(f)\|_{L_p(\mathcal{N})},$$
$$\|f\|_{\mathcal{H}_p^r}=\|S_r(f)\|_{L_p(\mathcal{N})}.$$

These are norms, which can be seen easily from the space $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$. So we define the spaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{R},\mathcal{M})$) as the completion of $(S_{\mathcal{N}},\|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}},\|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$. Now, we define the operator-valued Hardy spaces as follows: for $1\leq p<2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M})=\mathcal{H}_p^c(\mathbb{R},\mathcal{M})+\mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{0.2.3}$$

with the norm

$$\|f\|_{\mathcal{H}_p}=\inf\{\|g\|_{\mathcal{H}_p^c}+\|h\|_{\mathcal{H}_p^r}:f=g+h,g\in\mathcal{H}_p^c,h\in\mathcal{H}_p^r\}$$

and for $2\leq p<\infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M})=\mathcal{H}_p^c(\mathbb{R},\mathcal{M})\cap\mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{0.2.4}$$

with the norm defined as

$$\|f\|_{\mathcal{H}_p}=\max\{\|f\|_{\mathcal{H}_p^c},\|f\|_{\mathcal{H}_p^r}\}.$$

For $\varphi\in L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{\mathcal{BMO}^c}=\sup_{J\in\mathcal{D}}\Big\|\big(\frac{1}{|J|}\sum_{I\subset J}|\langle\varphi,w_I\rangle|^2\big)^{\frac12}\Big\|_{\mathcal{M}} \tag{0.2.5}$$

and

$$\|\varphi\|_{\mathcal{BMO}^r}=\|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

Define

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M})=\{\varphi\in L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dx}{1+x^2})):\|\varphi\|_{\mathcal{BMO}^c}<\infty\}$$

and

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^r} < \infty\}.$$

These are Banach spaces modulo constant functions. Now we define

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

As in the martingale case [43], we can also define $L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ for all $2 < p \leq \infty$. For $\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{L_p^c\mathcal{MO}} = \Big\|(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x}|\langle\varphi, w_I\rangle|^2)_k\Big\|_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)}^{\frac{1}{2}} \tag{0.2.6}$$

and

$$\|\varphi\|_{L_p^r\mathcal{MO}} = \|\varphi^*\|_{L_p^c\mathcal{MO}},$$

where I_k^x denote the unique dyadic interval with length 2^{-k+1} that containing x. We will use the convention adopted in [45] for the norm in $L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)$. Thus

$$\Big\|(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x}|\langle\varphi, w_I\rangle|^2)_k\Big\|_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} = \Big\|\sup_k{}^+\frac{1}{|I_k^x|}\sum_{I\subset I_k^x}|\langle\varphi, w_I\rangle|^2\Big\|_{L_{\frac{p}{2}}(\mathcal{N})}.$$

Again, we can define

$$L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^c\mathcal{MO}} < \infty\}$$

and

$$L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_r^c\mathcal{MO}} < \infty\}.$$

Define

$$L_p\mathcal{MO}(\mathbb{R},\mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \cap L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}).$$

Note that $L_\infty^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$.

Then we have following dual and interpolation theorem:

Theorem 0.2.1. *We have*

$$(\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \tag{0.2.7}$$

with equivalent norms.

Similarly, the duality holds between $\mathcal{H}_1^r$ and $\mathcal{BMO}^r$, between $\mathcal{H}_1$ and $\mathcal{BMO}$ with equivalent norms.

Theorem 0.2.2. *Let $1 < p < 2$. We have*

$$(\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^* = L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \tag{0.2.8}$$

with equivalent norms.

Similarly, the duality holds between $\mathcal{H}_p^r$ and $L_{p'}^r$, between $\mathcal{H}_p$ and $L_{p'}\mathcal{MO}$ with equivalent norms.

Theorem 0.2.3. *For any $1 < p < \infty$, we have*

$$(\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M}), \tag{0.2.9}$$

Theorem 0.2.4. *The following results hold with equivalent norms:*

(i) *Let* $1 \leq q < p < \infty$*, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} = L_p(\mathcal{N}). \tag{0.2.10}$$

(ii) *Let* $1 < q < p \leq \infty$*, we have*

$$[\mathcal{H}_1(\mathbb{R},\mathcal{M}), L_p(\mathcal{N})]_{\frac{p'}{q'}} = L_q(\mathcal{N}). \tag{0.2.11}$$

(iii) *Let* $1 < p < \infty$*, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_{\frac{1}{p}} = L_p(\mathcal{N}). \tag{0.2.12}$$

We denote the column Hardy space $H_p^c(\mathbb{R},\mathcal{M})$ and the bounded mean oscillation space $BMO^c(\mathbb{R},\mathcal{M})$ in [52]. We have the following result.

Theorem 0.2.5. *We have*

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = BMO^c(\mathbb{R},\mathcal{M})$$

with equivalent norms. Similar results holds for the row spaces. Consequently, $\mathcal{BMO}(\mathbb{R},\mathcal{M}) = BMO(\mathbb{R},\mathcal{M})$ *with equivalent norms.*

0.3 Chapter 3

Let $d \geq 2$ and $\theta = (\theta_{kj})$ be a real skew-symmetric $d \times d$-matrix. Recall that the d-dimensional noncommutative torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by d unitaries $U_1, \ldots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i}\theta_{kj}} U_j U_k, \quad j, k = 1, \ldots, d.$$

There exists a faithful tracial state τ on $\mathcal{A}_\theta$. Let $\mathbb{T}_\theta^d$ be the von Neumann algebra in the GNS representation of τ. $\mathbb{T}_\theta^d$ is called the quantum d-torus associated with θ. Note that if $\theta = 0$, then $\mathcal{A}_\theta = C(\mathbb{T}^d)$ and $\mathbb{T}_\theta^d = L_\infty(\mathbb{T}^d)$, where $\mathbb{T}^d$ is the usual d-torus. So a quantum d-torus is a deformation of the usual d-torus. It is thus natural to expect that $\mathbb{T}_\theta^d$ shares many properties with $\mathbb{T}^d$. This is indeed the case for differential geometry, as shown by the works of Connes and his collaborators. However, little is done regarding analysis. To our best knowledge, up to now, only the mean convergence theorem of quantum Fourier series by the square Fejér summation was proved at the C^*-algebra level (cf. [86, 87]), and on the other hand, the quantum torus analogue of Sobolev inequalities was obtained only in the Hilbert space case, or equivalently L_2 space case (cf. [80]). The reason might be explained by numerous difficulties one may encounter when dealing with noncommutative L_p-spaces. For instance, the usual way of proving pointwise convergence theorems is to pass through the corresponding maximal inequalities. But the study of maximal inequalities is one of the most subtle and difficult parts in noncommutative analysis.

This paper is the first one of a project that intends to develop analysis on quantum tori and more generally on twisted crossed products by amenable groups. Our aim here is to study some important aspects of harmonic analysis on $\mathbb{T}_\theta^d$. The subject that we address is three-fold:

i) *Convergence of Fourier series.* We will establish the maximal inequalities for various means of Fourier series defined on quantum tori and obtain the corresponding pointwise convergence theorems. In particular, we will prove the noncommutative analogue of the classical Stein theorem on Bochner-Riesz means.

ii) *Fourier multipliers.* We will prove that L_p $(1 < p \leq \infty)$ completely bounded Fourier multipliers on quantum tori coincide with those on classical tori with equivalent cb-norms.

iii) *Hardy and* BMO spaces. We will present the H_1-BMO and Littlewood-Paley theories associated with the Poisson semigroup over quantum tori.

One of main strategies for approaching these problems is to transfer them into the corresponding ones in the case of operator-valued functions on the classical tori. To state our main results we need some notation. Let $1 \leq p \leq \infty$ and $x \in L_p(\mathbb{T}^d_\theta)$. Then x admits a formal Fourier series:

$$x \sim \sum_{m\in\mathbb{Z}^d} \hat{x}(m)U^m, \tag{0.3.1}$$

where

$$\hat{x}(m) = \tau(x(U^m)^*),\ m \in \mathbb{Z}^d, \tag{0.3.2}$$

that is called the m-th *Fourier coefficient* of x. We remark that one of the main subjects of harmonic analysis on quantum tori, just as with $L_\infty(\mathbb{T}^d)$, is to study when and in what sense the series of the right hand side of (0.3.1) converges to x. As in the classical case, we will consider mainly three kinds of summation method:

1) The *square Cesàro mean*

$$F_N[x] = \sum_{\substack{m\in\mathbb{Z}^d,\\ |m|_\infty\leq N}} \Big(1 - \frac{|m_1|}{N+1}\Big)\cdots\Big(1 - \frac{|m_d|}{N+1}\Big)\hat{x}(m)U^m, \quad N \geq 0. \tag{0.3.3}$$

2) The *square Poisson mean*

$$P_r[x] = \sum_{m\in\mathbb{Z}^d} \hat{x}(m)r^{|m|_1}U^m, \quad 0 \leq r < 1. \tag{0.3.4}$$

3) The *circular Poisson mean*

$$\mathbb{P}_r[x] = \sum_{m\in\mathbb{Z}^d} \hat{x}(m)r^{|m|_2}U^m, \quad 0 \leq r < 1. \tag{0.3.5}$$

4) Let Φ be a continuous function on $\mathbb{R}^d$ with $\Phi(0) = 1$. Define

$$\Phi^\varepsilon[x] = \sum_{m\in\mathbb{Z}^d} \Phi(\varepsilon m)\hat{x}(m)U^m, \quad \varepsilon > 0.$$

We will always impose the following condition to Φ:

$$\begin{cases} \Phi(s) = \hat{\varphi}(s) \quad \text{with } \displaystyle\int_{\mathbb{R}^d} \varphi(s)ds = 1; \\ |\Phi(s)| + |\varphi(s)| \leq A(1 + |s|)^{-d-\delta},\ \forall s \in \mathbb{R}^d, \end{cases} \tag{0.3.6}$$

for some $A, \delta > 0$ (cf. [83, p. 253]).

Here, $|m|_p = (\sum_{j=1}^d |m_j|^p)^{1/p}$ for $1 \le p < \infty$, and $|m|_\infty = \sup_{1\le j\le d} |m_j|$.

The following is one of our main results:

Theorem 0.3.1. i) *Let $x \in L_1(\mathbb{T}_\theta^d)$. Then for any $\alpha > 0$ there exists a projection $e \in \mathbb{T}_\theta^d$ such that*

$$\sup_{N\ge 0} \big\|eF_N[x]e\big\|_\infty \le \alpha \quad \text{and} \quad \tau(e^\perp) \le C_d \, \frac{\|x\|_1}{\alpha}.$$

ii) *Let $1 < p \le \infty$. Then*

$$\big\|{\sup_{N\ge 0}}^+ F_N[x]\big\|_p \le C_d \, \frac{p^2}{(p-1)^2} \, \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}_\theta^d).$$

Both statements hold for the three other summation methods P_r, $\mathbb{P}_r$ and Φ^ε. In the case of Φ^ε, the constant C_d also depends on the two constants in (3.2.1).

As usual, the maximal inequalities in Theorem 0.3.1 should imply the corresponding pointwise convergence theorems. By adapting the arguments of M. Junge and Q. Xu [45], we indeed get the following result:

Theorem 0.3.2. *Let $1 \le p \le \infty$ and $x \in L_p(\mathbb{T}_\theta^d)$. Then $F_N[x] \xrightarrow{b.a.u.} x$ as $N \to \infty$. Moreover, for $2 \le p \le \infty$ the b.a.u. convergence can be strengthened to a.u. convergence.*

Similar statements hold for the two Poisson means P_r, $\mathbb{P}_r$ as $r \to \infty$ as well as for the mean Φ^ε as $\varepsilon \to 0$.

Here $x_n \xrightarrow{b.a.u} x$ means that (x_n) bilaterally almost uniformly (b.a.u, in short) converges to x, while $x_n \xrightarrow{a.u} x$ means almost uniform (a.u) convergence, that were both introduced by Lance [48] (see also [45] for the details).

One may also consider the *circular Cesàro mean*:

$$\mathbb{F}_N[x] = \sum_{\substack{m\in\mathbb{Z}^d,\\ |m|_2\le N}} \Big(1 - \frac{|m|_2}{N+1}\Big)\hat{x}(m)U^m, \quad N \ge 0. \tag{0.3.7}$$

However, the preceding results fail to hold for $\mathbb{F}_N$ even in the classical case [83]. Instead, one needs to consider the *Bochner-Riesz mean* of order α :

$$B_R^\alpha[x] = \sum_{|m|_2\le R} \Big(1 - \frac{|m|_2^2}{R^2}\Big)^\alpha \hat{x}(m)U^m. \tag{0.3.8}$$

If $\alpha > (d-1)/2$, the kernel of the Bochner-Riesz mean in this case is an approximation identity, we can then get the corresponding results associated with the mean convergence, maximal inequalities and pointwise convergence. In fact, both part (3) in Theorems 0.3.1 and 0.3.2, and the corresponding results concerning the *Bochner-Riesz mean* with the order α being greater than $(d-1)/2$, will be proved all as a consequence of the more general case concerning an approximation identity determined by a continuous function Φ on $\mathbb{R}^d$ with an additional condition on asymptotic behavior at infinite.

For the case $\alpha \le (d-1)/2$, we have the following theorem, which is the generalization of Stein's theorem [81] to the quantum tori.

Theorem 0.3.3. *Let $1 < p < \infty$ and $\alpha > (d-1)|\frac{1}{2} - \frac{1}{p}|$. Then*

(1) *For any* $x \in L_p(\mathbb{T}^d_\theta)$,

$$\big\|{\sup}^+_{R>0} B^\alpha_R[x]\big\|_p \leq C_p \|x\|_p.$$

(2) $\lim_{R\to\infty} B^\alpha_R[x] = x$ *in* $L_p(\mathbb{T}^d_\theta)$.

(3) *For any* $x \in L_p(\mathbb{T}^d_\theta)$, $B^\alpha_R[x] \xrightarrow{b.a.u} x$ *as* $R \to \infty$.

Now we turn to the second aspect of the theme in this chapter, discussing the completely bounded Fourier multipliers on the quantum tori. Our motivation arises from the classical Fourier multiplier theory on ordinary tori [20, 83].

We may define the Fourier multipliers on the quantum torus in an ordinary way. Let $\phi = (\phi_m)_{m\in\mathbb{Z}^d}$. We define T_ϕ by

$$\widehat{T_\phi x}(m) = \phi_m \hat{x}(m), \quad \forall m \in \mathbb{Z}^d,$$

for any $x \in \mathbb{T}^d_\theta$. We call ϕ a L_p bounded multiplier (resp. c.b. L_p multiplier) on the quantum torus $\mathbb{T}^d_\theta$, if the operator T_ϕ extends to a bounded (resp. c.b.) map from $L_p(\mathbb{T}^d_\theta)$ into $L_p(\mathbb{T}^d_\theta)$. Let $\mathrm{M}(L_p(\mathbb{T}^d_\theta))$ and $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$ denote the sets of all L_p bounded and completely bounded multipliers respectively. We have the following theorem:

Theorem 0.3.4. *Let* $1 < p \leq \infty$. *Then* $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta)) = \mathrm{M}_{\mathrm{cb}}(L_p(L_\infty(\mathbb{T}^d)))$ *with equivalent* cb-*norms.*

The third topic of this chapter deals with the H^1-BMO and Littlewood-Paley theories associated with the Poisson semigroup $\mathbb{P}_r$ on the quantum tori. For any $x \in \mathbb{T}^d_\theta$ define

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr\Big)^{1/2}.$$

For $1 \leq p < \infty$ let

$$\|x\|_{\mathrm{H}^c_p} = |\hat{x}(\mathbf{0})| + \|G_c(x)\|_{L_p(\mathbb{T}^d_\theta)}.$$

This is a norm on T^d_θ (cf. e.g. [37]). We define the column Hardy space $\mathrm{H}^c_p(\mathbb{T}^d_\theta)$ as the completion of $\mathbb{T}^d_\theta$ with respect to this norm. The row Hardy space $\mathrm{H}^r_p(\mathbb{T}^d_\theta)$ is defined to be the space of all x such that $x^* \in \mathrm{H}^c_p(\mathbb{T}^d_\theta)$ equipped with the natural norm. The mixture Hardy spaces are defined as follows: If $1 \leq p < 2$,

$$\mathrm{H}_p(\mathbb{T}^d_\theta) = \mathrm{H}^c_p(\mathbb{T}^d_\theta) + \mathrm{H}^r_p(\mathbb{T}^d_\theta)$$

equipped with the sum norm

$$\|x\|_{\mathrm{H}_p} = \inf\{\|G_c(a)\|_p + \|G_r(b)\| \,:\, x = a+b, a \in \mathrm{H}^c_p(\mathbb{T}^d_\theta), b \in \mathrm{H}^r_p(\mathbb{T}^d_\theta)\},$$

and if $2 \leq p < \infty$,

$$\mathrm{H}_p(\mathbb{T}^d_\theta) = \mathrm{H}^c_p(\mathbb{T}^d_\theta) \cap \mathrm{H}^r_p(\mathbb{T}^d_\theta)$$

equipped with the intersection norm

$$\|x\|_{\mathrm{H}_p} = \max\{\|G_c(x)\|_p,\ \|G_r(x)\|_p\}.$$

We will also study the BMO spaces over $\mathbb{T}^d_\theta$. Set

$$\mathrm{BMO}^c(\mathbb{T}^d_\theta) = \{x \in L_2(\mathbb{T}^d_\theta) \,:\, \sup_r \big\|\mathbb{P}_r\big[|x - \mathbb{P}_r[x]|^2\big]\big\|_\infty < \infty\}$$

equipped with the norm

$$\|x\|_{\mathrm{BMO}^c} = \max\big\{|\hat{x}(\mathbf{0})|,\ \sup_r \big\|\mathbb{P}_r\big[|x-\mathbb{P}_r[x]|^2\big]\big\|_\infty^{1/2}\big\}.$$

$\mathrm{BMO}^r(\mathbb{T}^d_\theta)$ is defined as the space of all x such that $x^* \in \mathrm{BMO}^c(\mathbb{T}^d_\theta)$ with the norm $\|x\|_{\mathrm{BMO}^r} = \|x^*\|_{\mathrm{BMO}^c}$. The mixture $\mathrm{BMO}(\mathbb{T}^d_\theta)$ is the intersection of these two spaces:

$$\mathrm{BMO}(\mathbb{T}^d_\theta) = \mathrm{BMO}^c(\mathbb{T}^d_\theta) \cap \mathrm{BMO}^r(\mathbb{T}^d_\theta)$$

with intersection the norm.

The above definitions are motivated by Hardy spaces of noncommutative martingales ([43, 69]) and of quantum Markov semigroups ([37, 38, 52]). The main results of this part are summarized in the following theorem which shows that the Hardy spaces on $\mathbb{T}^d_\theta$ possess the properties of the usual Hardy spaces, as expected.

Theorem 0.3.5. i) *Let $1 < p < \infty$. Then $\mathrm{H}_p(\mathbb{T}^d_\theta) = L_p(\mathbb{T}^d_\theta)$ with equivalent norms.*

ii) *The dual space of $\mathrm{H}^c_1(\mathbb{T}^d_\theta)$ is equal to $\mathrm{BMO}^c(\mathbb{T}^d_\theta)$ with equivalent norms via the duality bracket*

$$\langle x,\ y\rangle = \tau(xy^*), \quad x \in L_2(\mathbb{T}^d_\theta),\ y \in \mathrm{BMO}^c(\mathbb{T}^d_\theta).$$

The same assertion holds for the row and mixture spaces too.

iii) *Let $1 < p < \infty$. Then*

$$(\mathrm{BMO}^c(\mathbb{T}^d_\theta),\ \mathrm{H}^c_1(\mathbb{T}^d_\theta))_{1/p} = \mathrm{H}^c_p(\mathbb{T}^d_\theta) = (\mathrm{BMO}^c(\mathbb{T}^d_\theta),\ \mathrm{H}^c_1(\mathbb{T}^d_\theta))_{1/p,p}$$

with equivalent norms, where $(\,\cdot\,,\ \cdot\,)_{1/p}$ and $(\,\cdot\,,\ \cdot\,)_{1/p,p}$ denote respectively the complex and real interpolation functors.

iv) *Let $1 < p < \infty$ and $X_0 \in \{\mathrm{BMO}(\mathbb{T}^d_\theta),\ L_\infty(\mathbb{T}^d_\theta)\}$, $X_1 \in \{\mathrm{H}_1(\mathbb{T}^d_\theta),\ L_1(\mathbb{T}^d_\theta)\}$. Then*

$$(X_0,\ X_1)_{1/p} = L_p(\mathbb{T}^d_\theta) = (X_0,\ X_1)_{1/p,p}$$

with equivalent norms.

Chapter 1

Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales

Introduction

Atomic decomposition plays a fundamental role in the classical martingale theory and harmonic analysis. Atoms for martingales are usually defined in terms of stopping times. Unfortunately, the concept of stopping times is, up to now, not well-defined in the generic noncommutative setting (there are some works on this topic, see [2] and references therein). We note, however, that atoms can be defined without help of stopping times. Let us recall this in classical martingale theory. Given a probability space $(\Omega, \mathcal{F}, \mu)$, let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing filtration of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\cup_n \mathcal{F}_n\big)$ and let $(\mathcal{E}_n)_{n\geq 1}$ denote the corresponding family of conditional expectations. An $\mathcal{F}$-measurable function $a \in L_2$ is said to be an *atom* if there exist $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Such atoms are called *simple atoms* by Weisz [89] and are extensively studied by him (see [88] and [89]). Let us point out that atomic decomposition was first introduced in harmonic analysis by Coifman [12]. It is Herz [27] who initiated atomic decomposition for martingale theory. Recall that we denote by $\mathcal{H}_1(\Omega)$ the space of martingales f with respect to $(\mathcal{F}_n)_{n\geq 1}$ such that the quadratic variation $S(f) = \Big(\sum_n |df_n|^2\Big)^{1/2}$ belongs to $L_1(\Omega)$, and by $\mathsf{h}_1(\Omega)$ the space of martingales f such that the conditioned quadratic variation $s(f) = \Big(\sum_n \mathcal{E}_{n-1}|df_n|^2\Big)^{1/2}$ belongs to $L_1(\Omega)$. We say that a martingale $f = (f_n)_{n\geq 1}$ is predictable in L_1 if there exists an adapted sequence $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative functions such that $|f_n| \leq \lambda_{n-1}$ for all $n \geq 1$ and such that $\sup_n \lambda_n \in L_1(\Omega)$. We denote by $\mathcal{P}_1(\Omega)$ the space of all predictable martingales. In a disguised form in the proof of Theorem A_∞ in [27], Herz establishes an atomic description of $\mathcal{P}_1(\Omega)$. Since $\mathcal{P}_1(\Omega) = \mathcal{H}_1(\Omega)$ for regular martingales, this gives an atomic decomposition of $\mathcal{H}_1(\Omega)$ in the regular case. Such a decomposition is still valid in the general case but for $\mathsf{h}_1(\Omega)$ instead of $\mathcal{H}_1(\Omega)$, as shown by Weisz [88].

In this paper, we will present the noncommutative version of atoms and prove that atomic decomposition for the Hardy spaces of noncommutative martingales is valid for these atoms. Since there are two kinds of Hardy spaces, i.e., the column and row Hardy spaces in the noncommutative setting, we need to define the corresponding two type atoms. This is a main difference from the commutative case, but can be done by considering the right and left supports of martingales as being operators on Hilbert spaces. Roughly speaking, replacing the supports of atoms in the above (ii) by the right (resp. left) supports we obtain the concept of noncommutative right (resp. left) atoms, which are proved to be suitable for the column (resp. row) Hardy spaces. On the other hand, due to the noncommutativity some basic constructions based on stopping times for classical martingales are not valid in the noncommutative setting, our approach to the atomic decomposition for the conditioned Hardy spaces of noncommutative martingales is via the $\mathsf{h}_1 - \mathsf{bmo}$ duality. Recall that the duality equality $(\mathsf{h}_1)^* = \mathsf{bmo}$ was established independently by [34] and [61]. However, this method does not give an explicit atomic decomposition.

The other main result of this paper concerns the interpolation of the conditioned Hardy spaces h_p. Such kind of interpolation results involving Hardy spaces of noncommutative martingales first appear in Musat's paper [50] for the spaces $\mathcal{H}_p$. We will present an extension of these results to the conditioned case. Note that our method is much simpler and more elementary than Musat's arguments. It seems that even in the commutative case, our method is simpler than all existing approaches to the interpolation of Hardy spaces of martingales. The main idea is inspired by an equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$ introduced by Herz [28] in the commutative case. We translate this quasinorm to the noncommutative setting to obtain a new characterization of h_p, $0 < p \leq 2$, which is more convenient for interpolation. By this way we show that $(\mathsf{bmo}, \mathsf{h}_1)_{1/p} = \mathsf{h}_p$ for any $1 < p < \infty$.

The remainder of this paper is divided into four sections. In Section 1 we present some preliminaries and notation on the noncommutative L_p-spaces and various Hardy spaces of noncommutative martingales. The atomic decomposition of the conditioned Hardy space $\mathsf{h}_1(\mathcal{M})$ is presented in Section 2, from which we deduce the atomic decomposition of the Hardy space $\mathcal{H}_1(\mathcal{M})$ by Davis' decomposition. In Section 3 we define an equivalent quasinorm for $\mathsf{h}_p(\mathcal{M}), 0 < p \leq 2$, and discuss the description of the dual space of $\mathsf{h}_p(\mathcal{M}), 0 < p \leq 1$. Finally, using the results of Section 3, the interpolation results between bmo and h_1 are proved in Section 4.

Any notation and terminology not otherwise explained, are as used in [84] for theory of von Neumann algebras, and in [70] for noncommutative L_p-spaces. Also, we refer to a recent book by Xu [93] for an up-to-date exposition of theory of noncommutative martingales.

1.1 Preliminaries and notations

Throughout this paper, $\mathcal{M}$ will always denote a von Neumann algebra with a normal faithful normalized trace τ. For each $0 < p \leq \infty$, let $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ be the associated noncommutative L_p-spaces. We refer to [70] for more details and historical references on these spaces.

For $x \in L_p(\mathcal{M})$ we denote by $r(x)$ and $l(x)$ the right and left supports of x, respectively. Recall that if $x = u|x|$ is the polar decomposition of x, then $r(x) = u^*u$ and $l(x) = uu^*$. $r(x)$ (resp. $l(x)$) is also the least projection e such that $xe = x$ (resp. $ex = x$). If x is selfadjoint, $r(x) = l(x)$.

Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is w*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$.

A sequence $x = (x_n)$ in $L_1(\mathcal{M})$ is called a *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$.

If in addition, all x_n's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called an L_p-martingale. In this case we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called a bounded L_p-martingale.

Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $dx = (dx_n)$ is called the *martingale difference sequence* of x. x is called a *finite martingale* if there exists N such that $dx_n = 0$ for all $n \geq N$. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Let us now recall the definitions of the square functions and Hardy spaces for noncommutative martingales. Following [69], we introduce the column and row versions of square functions relative to a (finite) martingale $x = (x_n)$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

Let $1 \leq p < \infty$. Define $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) as the completion of all finite L_p-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). The Hardy space of noncommutative martingales is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\},$$

where the infimum is taken over all $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$ such that $x = y + z$. For $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

The reason that $\mathcal{H}_p(\mathcal{M})$ is defined differently according to $1 \leq p < 2$ or $2 \leq p \leq \infty$ is presented in [69]. In that paper Pisier and Xu prove the noncommutative Burkholder-Gundy inequalities which imply that $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ with equivalent norms for $1 < p < \infty$.

We now consider the conditioned version of $\mathcal{H}_p$ developed in [43]. Let $x = (x_n)_{n\geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2};$$

and

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}.$$

These will be called the column and row conditioned square functions, respectively. Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi) norm $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). For $p = \infty$, we define $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) as the Banach space of the $L_\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology.

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{if } 0 < p < \infty,$$

and

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty \quad \text{if } p = \infty.$$

Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.

We define the conditioned version of martingale Hardy spaces as follows: If $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M})$$

equipped with the (quasi) norm

$$\|x\|_{\mathsf{h}_p} = \inf\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r}\},$$

where the infimum is taken over all $w \in \mathsf{h}_p^d(\mathcal{M}), y \in \mathsf{h}_p^c(\mathcal{M})$ and $z \in \mathsf{h}_p^r(\mathcal{M})$ such that $x = w + y + z$. For $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p} = \max\{\|x\|_{\mathsf{h}_p^d}, \|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}\}.$$

The noncommutative Burkholder inequalities proved in [43] state that

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \tag{1.1.1}$$

with equivalent norms for all $1 < p < \infty$.

In the sequel, $(\mathcal{M}_n)_{n\geq 1}$ will be a filtration of von Neumann subalgebras of $\mathcal{M}$. All martingales will be with respect to this filtration.

1.2 Atomic decompositions

Let us now introduce the concept of noncommutative atoms.

Definition 1.2.1. $a \in L_2(\mathcal{M})$ is said to be a $(1, 2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \le \tau(e)^{-1/2}$.

Replacing (ii) by (ii)$'$ $l(a) \le e$, we get the notion of a $(1,2)_r$-atom.

Here, $(1,2)_c$-atoms and $(1,2)_r$-atoms are noncommutative analogues of $(1,2)$-atoms for classical martingales. In a later remark we will discuss the noncommutative analogue of $(p,2)$-atoms. These atoms satisfy the following useful estimates.

Proposition 1.2.2. *If a is a $(1,2)_c$-atom then*

$$\|a\|_{\mathcal{H}_1^c} \le 1 \quad \text{and} \quad \|a\|_{\mathsf{h}_1^c} \le 1.$$

The similar inequalities hold for $(1,2)_r$-atoms.

Proof. Let e be a projection associated with a satisfying (i) $-$ (iii) of Definition 1.2.1. Let $a_k = \mathcal{E}_k(a)$. Observe that $a_k = 0$ for $k \le n$, so $da_k = 0$ for $k \le n$. For $k \ge n+1$ we have

$$\begin{array}{rclcl} e|da_k|^2 & = & [\mathcal{E}_k(ea^*) - \mathcal{E}_{k-1}(ea^*)]da_k & = & |da_k|^2 \\ & = & da_k^*[\mathcal{E}_k(ae) - \mathcal{E}_{k-1}(ae)] & = & |da_k|^2 e. \end{array}$$

This gives

$$e|da_k|^2 = |da_k|^2 = |da_k|^2 e$$

for any $k \ge 1$. Hence, we obtain

$$eS_c(a) = S_c(a) = S_c(a)e.$$

Consequently, the noncommutative Hölder inequality implies

$$\|a\|_{\mathcal{H}_1^c} = \tau[eS_c(a)] \le \|S_c(a)\|_2\|e\|_2 = \|a\|_2\|e\|_2 \le 1.$$

Since $e \in \mathcal{M}_n$, for $k \ge n+1$ we have

$$\begin{array}{rclcl} e\mathcal{E}_{k-1}(|da_k|^2) & = & \mathcal{E}_{k-1}(e|da_k|^2) & = & \mathcal{E}_{k-1}(|da_k|^2) \\ & = & \mathcal{E}_{k-1}(|da_k|^2 e) & = & \mathcal{E}_{k-1}(|da_k|^2)e. \end{array}$$

Thus, we deduce

$$\|a\|_{\mathsf{h}_1^c} \le 1.$$

$\square$

Now, atomic Hardy spaces are defined as follows.

Definition 1.2.3. We define $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ as the Banach space of all $x \in L_1(\mathcal{M})$ which admit a decomposition

$$x = \sum_k \lambda_k a_k$$

with for each k, a_k a $(1,2)_c$-atom or an element in $L_1(\mathcal{M}_1)$ of norm ≤ 1, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}_1^{c,\mathrm{at}}} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of x described above.

Similarly, we define $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ and $\|\cdot\|_{\mathsf{h}_1^{r,\mathrm{at}}}$.

It is easy to see that $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ is a Banach space. By Proposition 1.2.2 we have the contractive inclusion $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_1^c(\mathcal{M})$. The following theorem shows that these two spaces coincide. That establishes the atomic decomposition of the conditioned Hardy space $\mathsf{h}_1^c(\mathcal{M})$. This is the main result of this section.

Theorem 1.2.4. *We have*

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \qquad \text{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}_1^c(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}_1^{c,\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1^c} \leq \|x\|_{\mathsf{h}_1^{c,\mathrm{at}}}.$$

Similarly, $\mathsf{h}_1^r(\mathcal{M}) = \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ *with the same equivalence constants.*

We will show the remaining inclusion $\mathsf{h}_1^c(\mathcal{M}) \subset \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ by duality. Recall that the dual space of $\mathsf{h}_1^c(\mathcal{M})$ is the space $\mathsf{bmo}^c(\mathcal{M})$ defined as follows (we refer to [34] and [61] for details). Let

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty < \infty\}$$

and equip $\mathsf{bmo}^c(\mathcal{M})$ with the norm

$$\|x\|_{\mathsf{bmo}^c} = \max\Big(\|\mathcal{E}_1(x)\|_\infty\ ,\ \sup_{n\geq 1}\|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\Big).$$

This is a Banach space. Similarly, we define the row version $\mathsf{bmo}^r(\mathcal{M})$. Since $x_n = \mathcal{E}_n(x)$, we have

$$\mathcal{E}_n|x - x_n|^2 = \mathcal{E}_n|x|^2 - |x_n|^2 \leq \mathcal{E}_n|x|^2.$$

Thus the contractivity of the conditional expectation yields

$$\|x\|_{\mathsf{bmo}^c} \leq \|x\|_\infty. \tag{1.2.1}$$

We will describe the dual space of $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ as a noncommutative Lipschitz space defined as follows. We set

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

with

$$\|x\|_{\Lambda^c} = \max\Big(\|\mathcal{E}_1(x)\|_\infty\ ,\ \sup_{n\geq 1}\sup_{e\in\mathcal{P}_n}\tau(e)^{-1/2}\tau\big(e|x - x_n|^2\big)^{1/2}\Big),$$

where $\mathcal{P}_n$ denotes the lattice of projections of $\mathcal{M}_n$. Similarly, we define

$$\Lambda^r(\mathcal{M}) = \{x \in L^2(\mathcal{M}) : x^* \in \Lambda^c(\mathcal{M})\}$$

equipped with the norm

$$\|x\|_{\Lambda^r} = \|x^*\|_{\Lambda^c}.$$

The relation between Lipschitz space and bmo space can be stated as follows.

Proposition 1.2.5. *We have* $\mathsf{bmo}^c(\mathcal{M}) = \Lambda^c(\mathcal{M})$ *and* $\mathsf{bmo}^r(\mathcal{M}) = \Lambda^r(\mathcal{M})$ *isometrically.*

Proof. Let $x \in \mathsf{bmo}^c(\mathcal{M})$. It is obvious that by the noncommutative Hölder inequality we have, for all $n \geq 1$,

$$\sup_{e \in \mathcal{P}_n} \tau(e)^{-1/2} \tau\big(e|x - x_n|^2\big)^{1/2} \leq \|\mathcal{E}_n |x - x_n|^2\|_\infty^{1/2}.$$

To prove the reverse inclusion, by duality we can write

$$\begin{aligned} \|\mathcal{E}_n |x - x_n|^2\|_\infty &= \sup_{\|y\|_1 \leq 1,\, y \in L_1^+(\mathcal{M}_n)} \big|\tau(y|x - x_n|^2)\big| \\ &= \sup_{e \in \mathcal{P}_n} \tau(e)^{-1} \tau(e|x - x_n|^2), \end{aligned}$$

where the last equality comes from the density of linear combinations of mutually disjoint projections in $L_1(\mathcal{M}_n)$. Thus $\|x\|_{\Lambda^c} = \|x\|_{\mathsf{bmo}^c}$, and the same holds for the row spaces. □

We now turn to the duality between the conditioned atomic space $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ and the Lipschitz space $\Lambda^c(\mathcal{M})$.

Theorem 1.2.6. *We have $h_1^{c,\mathrm{at}}(\mathcal{M})^* = \Lambda^c(\mathcal{M})$ isometrically. More precisely,*

(i) *Every $x \in \Lambda^c(\mathcal{M})$ defines a continuous linear functional on $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ by*

$$\varphi_x(y) = \tau(x^* y), \quad \forall y \in L_2(\mathcal{M}). \tag{1.2.2}$$

(ii) *Conversely, each $\varphi \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})^*$ is given as* (1.2.2) *by some $x \in \Lambda^c(\mathcal{M})$.*

Similarly, $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})^* = \Lambda^r(\mathcal{M})$ isometrically.

Remark 1.2.7. *Remark that we have defined the duality bracket* (1.2.2) *for operators in $L_2(\mathcal{M})$. This is sufficient for $L_2(\mathcal{M})$ is dense in $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$. The latter density easily follows from the decomposition $L_2(\mathcal{M}) = L_2^0(\mathcal{M}) \oplus L_2(\mathcal{M}_1)$, where $L_2^0(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \mathcal{E}_1(x) = 0\}$.*

Proof of Theorem 1.2.6. We first show $\Lambda^c(\mathcal{M}) \subset h_1^{c,\mathrm{at}}(\mathcal{M})^*$. In fact we will not need this inclusion for the proof of Theorem 1.2.4, however we include the proof for the sake of completeness. Let $x \in \Lambda^c(\mathcal{M})$. For any $(1,2)_c$-atom a associated with a projection e satisfying (i) – (iii) of Definition 1.2.1, by the noncommutative Hölder inequality we have

$$\begin{aligned} \big|\tau(x^* a)\big| &= \big|\tau((x - x_n)^* a e)\big| \\ &\leq \|e(x - x_n)^*\|_2 \|a\|_2 \\ &\leq \tau(e)^{-1/2} \big[\tau(e|x - x_n|^2)\big]^{1/2} \\ &\leq \|x\|_{\Lambda^c}. \end{aligned}$$

On the other hand, for any $a \in L_1(\mathcal{M}_1)$ with $\|a\|_1 \leq 1$ we have

$$|\tau(x^* a)| = |\tau(\mathcal{E}_1(x)^* a)| \leq \|\mathcal{E}_1(x)\|_\infty \|a\|_1 \leq \|x\|_{\Lambda^c}.$$

Thus, we deduce that

$$\big|\tau(x^* y)\big| \leq \|x\|_{\Lambda^c} \|y\|_{\mathsf{h}_1^{c,\mathrm{at}}}$$

for all $y \in L_2(\mathcal{M})$. Hence, φ_x extends to a continuous functional on $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ of norm less than or equal to $\|x\|_{\Lambda^c}$.

Conversely, let $\varphi \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})^*$. As explained in the previous remark, $L_2(\mathcal{M}) \subset \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ so by the Riesz representation theorem there exists $x \in L_2(\mathcal{M})$ such that

$$\varphi(y) = \tau(x^* y), \quad \forall y \in L_2(\mathcal{M}).$$

Fix $n \geq 1$ and let $e \in \mathcal{P}_n$. We set

$$y_e = \frac{(x - x_n)e}{\|(x - x_n)e\|_2 \tau(e)^{1/2}}.$$

It is clear that y_e is a $(1,2)_c$-atom with the associated projection e. Then

$$\|\varphi\| \geq |\varphi(y_e)| = |\tau((x - x_n)^* y_e)| = \frac{1}{\tau(e)^{1/2}} \left[\tau(e|x - x_n|^2)\right]^{1/2}.$$

On the other hand, let $y \in L_1(\mathcal{M}_1), \|y\|_1 \leq 1$ be such that $\|\mathcal{E}_1(x)\|_\infty = |\tau(x^* y)|$. Then $\|\mathcal{E}_1(x)\|_\infty \leq \|\varphi\|$. Combining these estimates we obtain $\|x\|_{\Lambda^c} \leq \|\varphi\|$. This ends the proof of the duality $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = \Lambda^c(\mathcal{M})$. Passing to adjoints yields the duality $(\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}))^* = \Lambda^r(\mathcal{M})$. □

We can now prove the reverse inclusion of Theorem 1.2.4.

Proof of Theorem 1.2.4. By Proposition 1.2.2 we already know that $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_1^c(\mathcal{M})$. Combining Proposition 1.2.5 and Theorem 1.2.6 we obtain that $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = \mathsf{bmo}^c(\mathcal{M})$ with equal norms. The duality between $\mathsf{h}_1^c(\mathcal{M})$ and $\mathsf{bmo}^c(\mathcal{M})$ proved in [34] and [61] then yields that $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = (\mathsf{h}_1^c(\mathcal{M}))^*$ with the following equivalence constants

$$\frac{1}{\sqrt{2}} \|\varphi_x\|_{(\mathsf{h}_1^c)^*} \leq \|x\|_{\mathsf{bmo}^c} = \|\varphi_x\|_{(\mathsf{h}_1^{c,\mathrm{at}})^*} \leq \|\varphi_x\|_{(\mathsf{h}_1^c)^*}.$$

This ends the proof of Theorem 1.2.4. □

We can generalize this decomposition to the whole space $\mathsf{h}_1(\mathcal{M})$. To this end we need the following definition.

Definition 1.2.8. We set

$$\mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_1^d(\mathcal{M}) + \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}),$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_1^{\mathrm{at}}} = \inf\left\{\|w\|_{\mathsf{h}_1^d} + \|y\|_{\mathsf{h}_1^{c,\mathrm{at}}} + \|z\|_{\mathsf{h}_1^{r,\mathrm{at}}}\right\},$$

where the infimum is taken over all $w \in \mathsf{h}_1^d(\mathcal{M}), y \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$, and $z \in \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ such that $x = w + y + z$.

Thus Theorem 1.2.4 clearly implies the following.

Theorem 1.2.9. *We have*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}_1(\mathcal{M})$

$$\frac{1}{\sqrt{2}} \|x\|_{\mathsf{h}_1^{\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathsf{h}_1^{\mathrm{at}}}.$$

The noncommutative Davis' decomposition presented in [61] state that $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})$. Thus Theorem 1.2.9 yields that $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$, which means that we can decompose any martingale in $\mathcal{H}_1(\mathcal{M})$ in an atomic part and a diagonal part. This is the atomic decomposition for the Hardy space of noncommutative martingales.

Remark 1.2.10. At the time of this writing, we do not know how to construct the above atomic decompositions explicitly. One encounters some substantial difficulties in trying to adapt the classical atomic constructions to the noncommutative setting.

Problem 1.2.11. Find a constructive proof of Theorem 1.2.4 or Theorem 1.2.9.

To end this section we discuss the case of h_p for $0 < p < 1$. We define the noncommutative analogue of $(p, 2)$-atoms as follows.

Definition 1.2.12. Let $0 < p \leq 1$. $a \in L_2(\mathcal{M})$ is said to be a $(p, 2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{1/2-1/p}$.

Replacing (ii) by (ii)$'$ $l(a) \leq e$, we get the notion of a $(p, 2)_r$-atom.

We define $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ and $\mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ as in Definition 1.2.3. As for $p = 1$, we have $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M})$ contractively.

On the other hand, we can describe the dual space of $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ as a Lipschitz space. For $\alpha \geq 0$, we set

$$\Lambda_\alpha^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda_\alpha^c} < \infty\}$$

with

$$\|x\|_{\Lambda_\alpha^c} = \sup_{n\geq 1} \sup_{e\in\mathcal{P}_n} \tau(e)^{-1/2-\alpha} \tau\big(e|x - x_n|^2\big)^{1/2}.$$

By a slight modification of the proof of Theorem 1.2.6 (by setting $y_e = \frac{(x-x_n)e}{\|(x-x_n)e\|_2 \tau(e)^{1/2-1/p}}$) we can show that $(\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}))^* = \Lambda_\alpha^c(\mathcal{M})$ for $0 < p \leq 1$, with $\alpha = 1/p - 1$.

At the time of this writing we do not know if $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ coincides with $\mathsf{h}_p^c(\mathcal{M})$. The problem of the atomic decomposition of $\mathsf{h}_p(\mathcal{M})$ for $0 < p < 1$ is entirely open, and is related to Problem 1.2.11. We record this problem explicitly here:

Problem 1.2.13. *Does one have $\mathsf{h}_p^c(\mathcal{M}) = \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ for $0 < p < 1$?*

1.3 An equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$

In the commutative case Herz described in [28] an equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$. This section is devoted to determining a noncommutative analogue of this. This characterization of h_p will be useful in the sequel. Indeed, this will imply an interpolation result in the next section. To define equivalent quasinorms of $\|\cdot\|_{\mathsf{h}_p^c}$ and $\|\cdot\|_{\mathsf{h}_p^r}$ for $0 < p \leq 2$ we introduce the index class W which consists of sequences $\{w_n\}_{n\in\mathcal{N}}$ such that $\{w_n^{2/p-1}\}_{n\in\mathcal{N}}$ is nondecreasing with each $w_n \in L_1^+(\mathcal{M}_n)$ invertible with bounded inverse and $\|w_n\|_1 \leq 1$.

For an L_2-martingale x we set

$$N_p^c(x) = \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}|^2\Big)\Big]^{1/2}$$

and

$$N_p^r(x) = \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}^*|^2\Big)\Big]^{1/2}.$$

We need the following well-known lemma, and include a proof for the convenience of the reader.

Lemma 1.3.1. *Let f be a function in $C^1(\mathbb{R}^+)$ and $x, y \in \mathcal{M}^+$. Then*

$$\tau(f(x+y) - f(x)) = \tau\Big(\int_0^1 f'(x+ty)ydt\Big).$$

Proof. We set $\varphi(t) = \tau(f(x+ty))$, for $t \in [0,1]$. Then $\varphi'(0) = \tau(f'(x)y)$. Indeed, the tracial property of τ implies this equality for $f(t) = t^n, n \in \mathcal{N}$. We can extend this result for all f polynomial by linearity, then for all f by approximation. A translation argument gives $\varphi'(t) = \tau(f'(x+ty)y), \forall t \in [0,1]$. Writing $\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)dt$ we obtain the required result. □

Proposition 1.3.2. *For $0 < p \leq 2$ and $x \in L_2(\mathcal{M})$ we have*

$$\Big(\frac{p}{2}\Big)^{1/2} N_p^c(x) \leq \|x\|_{\mathsf{h}_p^c} \leq N_p^c(x). \tag{1.3.1}$$

A similar statement holds for $\mathsf{h}_p^r(\mathcal{M})$ and N_p^r.

Proof. Note that

$$\begin{aligned} N_p^c(x) &= \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}\mathcal{E}_n|dx_{n+1}|^2\Big)\Big]^{1/2} \\ &= \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)\Big)\Big]^{1/2}. \end{aligned}$$

Let $x \in L_2(\mathcal{M})$ with $\|x\|_{\mathsf{h}_p^c} < 1$. By approximation we can assume that $x \in L_\infty(\mathcal{M})$ and $s_{c,n}(x)$ is invertible with bounded inverse for every $n \geq 1$. Then $\{s_{c,n+1}(x)^p\} \in W$; so

$$N_p^c(x) \leq \Big[\tau\Big(\sum_{n\geq 0} s_{c,n+1}(x)^{p-2}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)\Big)\Big]^{1/2}.$$

Applying Lemma 1.3.1 with $f(t) = t^{p/2}, x+y = s_{c,n+1}(x)^2$ and $x = s_{c,n}(x)^2$ we obtain

$$\begin{aligned} &\tau(s_{c,n+1}(x)^p - s_{c,n}(x)^p) = \\ &\tau\Big(\int_0^1 \frac{p}{2}\big[s_{c,n}(x)^2 + t(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)\big]^{\frac{p}{2}-1}\big[s_{c,n+1}(x)^2 - s_{c,n}(x)^2\big]dt\Big) \\ &\geq \frac{p}{2}\tau(s_{c,n+1}(x)^{p-2}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)), \end{aligned}$$

where we have used the fact that the operator function $a \mapsto a^{\frac{p}{2}-1}$ is nonincreasing for $-1 < \frac{p}{2} - 1 \leq 0$. Taking the sum over n leads to

$$N_p^c(x)^2 \leq \frac{2}{p}\tau(s_c(x)^p) = \frac{2}{p}.$$

We turn to the other estimate. Given $\{w_n\} \in W$ put

$$w^{2/p-1} = \lim_{n\to+\infty} w_n^{2/p-1} = \sup_n w_n^{2/p-1}.$$

It follows that $\{w_n^{1-2/p}\}$ decreases to $w^{1-2/p}$ and

$$\begin{aligned}\tau\Big(\sum_{n\ge 0} w_n^{1-2/p}|dx_{n+1}|^2\Big) &\ge \tau\Big(w^{1-2/p}\sum_{n\ge 0}\mathcal{E}_n|dx_{n+1}|^2\Big)\\ &= \tau\Big(w^{1-2/p}s_c(x)^2\Big).\end{aligned}$$

Since $\frac{1}{p} = \frac{1}{2} + \frac{2-p}{2p}$ the Hölder inequality gives

$$\begin{aligned}\|s_c(x)\|_p &= \|w^{1/p-1/2}w^{1/2-1/p}s_c(x)\|_p\\ &\le \|w^{1/p-1/2}\|_{2p/(2-p)}\|w^{1/2-1/p}s_c(x)\|_2\\ &= \tau(w)^{1/p-1/2}\tau(w^{1-2/p}s_c(x)^2)^{1/2}.\end{aligned}$$

Now $\tau(w) \le 1$; so we have

$$\|s_c(x)\|_p \le \Big[\tau\Big(\sum_{n\ge 0} w_n^{1-2/p}|dx_{n+1}|^2\Big)\Big]^{1/2}$$

for all $\{w_n\} \in W$. □

Thus the quasinorm N_p^c is equivalent to $\|\cdot\|_{\mathsf{h}_p^c}$ on $L_2(\mathcal{M})$. So $\mathsf{h}_p^c(\mathcal{M})$ can also be defined as the completion of all finite L_2-martingales with respect to N_p^c for $0 < p \le 2$. This new characterization of $\mathsf{h}_p^c(\mathcal{M})$ yields the following description of its dual space.

Theorem 1.3.3. *Let $0 < p \le 2$ and $\frac{1}{q} = 1 - \frac{1}{p}$. Then the dual space of $\mathsf{h}_p^c(\mathcal{M})$ coincide with the L_2-martingales x for which $M_q^c(x) = \sup_W \Big[\tau\Big(\sum_{n\ge 0} w_n^{1-2/q}|dx_{n+1}|^2\Big)\Big]^{1/2} < \infty$. More precisely,*

(i) *Every L_2-martingale x such that $M_q^c(x) < \infty$ defines a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ by*
$$\phi_x(y) = \tau(yx^*) \text{ for } y \in L_2(\mathcal{M}).$$

(ii) *Conversely, any continuous linear functional ϕ on $\mathsf{h}_p^c(\mathcal{M})$ is given as above by some x such that $M_q^c(x) < \infty$.*

Similarly, the dual space of $\mathsf{h}_p^r(\mathcal{M})$ coincide with the L_2-martingales x for which $M_q^r(x) = M_q^c(x^*) < \infty$.

Proof. Let x be such that $M_q^c(x) < \infty$. Then x defines a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ by $\phi_x(y) = \tau(yx^*)$ for $y \in L_2(\mathcal{M})$. To see this fix $\{w_n\} \in W$. The Cauchy-Schwarz inequality gives

$$\begin{aligned}\tau(yx^*) &= \sum_{n\ge 0}\tau\Big((dy_{n+1}w_n^{1/2-1/p})(dx_{n+1}w_n^{1/2-1/q})^*\Big)\\ &\le \Big(\sum_{n\ge 0}\tau(w_n^{1-2/p}|dy_{n+1}|^2)\Big)^{1/2}\Big(\sum_{n\ge 0}\tau(w_n^{1-2/q}|dx_{n+1}|^2)\Big)^{1/2}\\ &\le \Big(\sum_{n\ge 0}\tau(w_n^{1-2/p}|dy_{n+1}|^2)\Big)^{1/2} M_q^c(x).\end{aligned}$$

Taking the infimum over W we obtain $\tau(yx^*) \leq N_p^c(y)M_q^c(x)$.

Conversely, let ϕ be a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ of norm ≤ 1. As $L_2(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M})$, ϕ induces a continuous linear functional on $L_2(\mathcal{M})$. Thus there exists $x \in L_2(\mathcal{M})$ such that $\phi(y) = \tau(yx^*)$ for $y \in L_2(\mathcal{M})$. By the density of $L_2(\mathcal{M})$ in $\mathsf{h}_p^c(\mathcal{M})$ we have

$$\|\phi\|_{(\mathsf{h}_p^c)^*} = \sup_{y\in L_2(\mathcal{M}), \|y\|_{\mathsf{h}_p^c}\leq 1} |\tau(yx^*)| \leq 1.$$

Thus by Proposition 1.3.2 we obtain

$$\sup_{y\in L_2(\mathcal{M}), N_p^c(y)\leq 1} |\tau(yx^*)| \leq 1. \tag{1.3.2}$$

We want to show that $M_q^c(x) < \infty$. Fix $\{w_n\} \in W$. Let y be the martingale defined by $dy_{n+1} = dx_{n+1}w_n^{1-2/q}, \forall n \in \mathcal{N}$. By (1.3.2) we have

$$\begin{aligned} \tau(yx^*) &= \tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big) \leq N_p^c(y) \\ &\leq \tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big)^{1/2}. \end{aligned}$$

Thus

$$\tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big) \leq 1, \quad \forall \{w_n\} \in W.$$

Taking the supremum over W we obtain $M_q^c(x) \leq 1$.

Passing to adjoints yields the description of the continuous linear functionals on $\mathsf{h}_p^r(\mathcal{M})$. $\square$

Remark that for $-\infty < 1/q \leq 1/2$, M_q^c and M_q^r define two norms. Let X_q^c (resp. X_q^r) be the Banach space consisting of the L_2-martingales x for which $M_q^c(x)$ (resp. $M_q^r(x)$) is finite. Theorem 1.3.3 shows that $(\mathsf{h}_p^c(\mathcal{M}))^* = X_q^c$ and $(\mathsf{h}_p^r(\mathcal{M}))^* = X_q^r$ for $0 < p \leq 2$, $\frac{1}{q} = 1 - \frac{1}{p}$.

For $-\infty < 1/q \leq 1/2$, note that $M_q^c(x)$ can be rewritten in the following form. Given $\{w_n\}_{n\geq 0} \in W$ we put

$$g_n = (w_n^{2/s} - w_{n-1}^{2/s})^{1/2}, \quad \forall n \geq 1$$

where $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It is clear that

$$\{g_n\}_{n\geq 1} \in G = \Big\{\{h_n\}_{n\geq 1}; h_n \in L_s(\mathcal{M}_n), \tau\Big(\Big(\sum_{n\geq 1} |h_n|^2\Big)^{s/2}\Big) \leq 1\Big\}.$$

Then

$$M_q^c(x) = \sup_G \Big[\tau\Big(\sum_{n\geq 1} |g_n|^2\mathcal{E}_n|x - x_n|^2\Big)\Big]^{1/2}.$$

It is now easy to see that the dual form of Junge's noncommutative Doob maximal inequality ([35]) implies that for $q \geq 2$, $X_q^c = L_q^c\mathsf{mo}(\mathcal{M})$ with equivalent norms, where $L_q^c\mathsf{mo}(\mathcal{M})$ is defined in [61].

Similarly, we have $X_q^r = L_q^r\mathsf{mo}(\mathcal{M})$ with equivalent norms.

Thus for $1 \leq p \leq 2$, Theorem 1.3.3 gives another proof of the duality obtained in [61] between $\mathsf{h}_p(\mathcal{M})$ and $L_q\mathsf{mo}(\mathcal{M})$ for $\frac{1}{p} + \frac{1}{q} = 1$. Note that this new proof is much simpler and yields a better constant for the upper estimate, that is $\sqrt{p/2}$ instead of $\sqrt{2}$.

For $0 < p < 1$, Theorem 1.3.3 leads to a first description of the dual space of $\mathsf{h}_p(\mathcal{M})$. However, this description is not satisfactory. Following the classical case, we would like to describe this dual space as the Lipschitz space $\Lambda^c_\alpha(\mathcal{M})$ defined in the previous section as the dual space of $\mathsf{h}^{c,\mathrm{at}}_p(\mathcal{M})$. Thus the description of the dual space of $\mathsf{h}_p(\mathcal{M})$ for $0 < p < 1$ is closely related to the atomic decomposition of $\mathsf{h}_p(\mathcal{M})$.

1.4 Interpolation of h_p spaces

It is a rather easy matter to identify interpolation spaces between commutative or noncommutative L_p-spaces by real or complex method. However, we need more efforts to establish interpolation results between Hardy spaces of martingales (see [32], and also [94]). Musat ([50]) extended Janson and Jones' interpolation theorem for Hardy spaces of martingales to the noncommutative setting. She proved in particular that for $1 \leq q < q_\theta < \infty$

$$(\mathcal{BMO}^c(\mathcal{M}), \mathcal{H}^c_q(\mathcal{M}))_{\frac{q}{q_\theta}} = \mathcal{H}^c_{q_\theta}(\mathcal{M}). \tag{1.4.1}$$

See also [36] for a different proof with better constants. This section is devoted to showing the analogue of (1.4.1) in the conditioned case. Our approach is simpler and more elementary than Musat's and also valid for her situation.

We refer to [8] for details on interpolation. Recall that the noncommutative L_p-spaces associated with a semifinite von Neumann algebra form interpolation scales with respect to the complex method and the real method. More precisely, for $0 < \theta < 1, 1 \leq p_0 < p_1 \leq \infty$ and $1 \leq q_0, q_1, q \leq \infty$ we have

$$L_p(\mathcal{M}) = (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta \quad \text{(with equal norms)} \tag{1.4.2}$$

and

$$L_{p,q}(\mathcal{M}) = (L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M}))_{\theta,q} \quad \text{(with equivalent norms)} \tag{1.4.3}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and where $L_{p,q}(\mathcal{M})$ denotes the noncommutative Lorentz space on $(\mathcal{M}, \tau)$.

We can now state the main result of this section which deals with complex interpolation between the column spaces $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{h}^c_1(\mathcal{M})$.

Theorem 1.4.1. *Let $1 < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}^c_1(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}^c_p(\mathcal{M}). \tag{1.4.4}$$

Remark 1.4.2. All spaces considered here are compatible in the sense that they can be embedded in the $*$-algebra of measurable operators with respect to $(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathcal{N}^2)), \tau\otimes\mathrm{Tr})$. Indeed, for each $1 \leq p < \infty$, $\mathsf{h}^c_p(\mathcal{M})$ can be identified with a subspace of $L_p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathcal{N}^2)))$. Recall that $\mathsf{h}^c_p(\mathcal{M})$ is also defined as the closure in $L^{\mathrm{cond}}_p(\mathcal{M}; \ell^c_2)$ of all finite martingale differences in $\mathcal{M}$. Here $L^{\mathrm{cond}}_p(\mathcal{M}; \ell^c_2)$ is the subspace of $L_p(\mathcal{M}, \ell^c_2(\mathcal{N}^2))$ introduced by Junge [35] consisting of all double indexed sequences (x_{nk}) such that $x_{nk} \in L_p(\mathcal{M}_n)$ for all $k \in \mathcal{N}$. We refer to [69] for details on the column and row spaces $L_p(\mathcal{M}, \ell^c_2)$ and $L_p(\mathcal{M}, \ell^r_2)$. Furthermore, by the Hölder inequality and duality, recalling that the trace is finite, we have, for $1 \leq p < q < \infty$, the continuous inclusions

$$L_\infty(\mathcal{M}) \subset \mathsf{bmo}^c(\mathcal{M}) \subset \mathsf{h}^c_q(\mathcal{M}) \subset \mathsf{h}^c_p(\mathcal{M}).$$

The first inclusion is proved by (1.2.1). The second one comes from the third one by duality. Indeed, it is proved in [43] that for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we have $(\mathsf{h}^c_p(\mathcal{M}))^* = \mathsf{h}^c_{p'}(\mathcal{M})$, and, as already mentioned above, we have $(\mathsf{h}_1^c(\mathcal{M}))^* = \mathsf{bmo}^c(\mathcal{M})$ (see [61]). Note that $L_\infty(\mathcal{M})$ is dense in all spaces above, except $\mathsf{bmo}^c(\mathcal{M})$. This implies that $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{h}_q^c(\mathcal{M})$ are dense in $\mathsf{h}_p^c(\mathcal{M})$ for $1 \leq p < q < \infty$.

We will need Wolff's interpolation theorem (see [90]). This result states that given Banach spaces E_i $(i = 1, 2, 3, 4)$ such that $E_1 \cap E_4$ is dense in both E_2 and E_3, and

$$E_2 = (E_1, E_3)_\theta \quad \text{and} \quad E_3 = (E_2, E_4)_\phi$$

for some $0 < \theta, \phi < 1$, then

$$E_2 = (E_1, E_4)_\varsigma \quad \text{and} \quad E_3 = (E_1, E_4)_\xi, \tag{1.4.5}$$

where $\varsigma = \frac{\theta\phi}{1-\theta+\theta\phi}$ and $\xi = \frac{\phi}{1-\theta+\theta\phi}$. The main step of the proof of Theorem 1.4.1 is the following lemma which is based on the equivalent quasinorm N_p^c of $\|\cdot\|_{\mathsf{h}_p^c}$ described in the previous section.

Lemma 1.4.3. *Let $1 < p < \infty$ and $0 < \theta < 1$. Then, the following holds with equivalent norms*

$$(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta = \mathsf{h}_q^c(\mathcal{M}), \tag{1.4.6}$$

where $\frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{q}$.

Proof. **Step 1:** We first prove (1.4.6) in the case $1 < q < p \leq 2$. As explained in Remark 1.4.2, $\mathsf{h}_p^c(\mathcal{M})$ can be identified with a subspace of
$L_p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathcal{N}^2)))$. Thus the interpolation between noncommutative L_p-spaces in (1.4.2) gives the inclusion $(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta \subset \mathsf{h}_q^c(\mathcal{M})$.

The reverse inclusion needs more efforts. This can be shown using the equivalent quasinorm N_p^c of $\|\cdot\|_{\mathsf{h}_p^c}$ defined previously. Let x be an L_2-finite martingale such that $\|x\|_{\mathsf{h}_q^c} < 1$. By (1.3.1) we have

$$N_q^c(x) = \inf_W \Big[\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big)\Big]^{1/2} < \Big(\frac{2}{q}\Big)^{1/2}.$$

Let $\{w_n\} \in W$ be such that

$$\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big) < \frac{2}{q}. \tag{1.4.7}$$

For $\varepsilon > 0$ and $z \in S$ we define

$$\begin{aligned} f_\varepsilon(z) &= \exp(\varepsilon(z^2-\theta^2))\sum_n dx_{n+1} w_n^{\frac{1}{2}-\frac{1}{q}} w_n^{\frac{1-z}{1}+\frac{z}{p}-\frac{1}{2}} \\ &= \exp(\varepsilon(z^2-\theta^2))\sum_n dx_{n+1} w_n^{1-(1-\frac{1}{p})z-\frac{1}{q}}. \end{aligned}$$

Then f_ε is continuous on S, analytic on S_0 and $f_\varepsilon(\theta) = x$. The term $\exp(\varepsilon(z^2-\theta^2))$ ensure that $f_\varepsilon(it)$ and $f_\varepsilon(1+it)$ tend to 0 as t goes to infinity. A direct computation gives for all $t \in \mathbb{R}$

$$\tau\Big(\sum_n w_n^{-1}|d(f_\varepsilon)_{n+1}(it)|^2\Big) = \exp(-2\varepsilon(t^2+\theta^2))\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big).$$

By (1.4.7) and (1.3.1) we obtain

$$\|f_\varepsilon(it)\|_{\mathsf{h}_1^c} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2}.$$

Similarly,

$$\|f_\varepsilon(1+it)\|_{\mathsf{h}_p^c} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2}.$$

Thus $x = f_\varepsilon(\theta) \in (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$ and

$$\|x\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2};$$

whence

$$\|x\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \leq \Big(\frac{2}{q}\Big)^{1/2}\|x\|_{\mathsf{h}_q^c}.$$

Step 2: To obtain the general case, we use Wolff's interpolation theorem mentioned above. Let us first recall that for $1 < v, s, q < \infty$ and $0 < \eta < 1$ such that $\frac{1}{q} = \frac{1-\eta}{v} + \frac{\eta}{s}$, we have with equivalent norms

$$(\mathsf{h}_v^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_\eta = \mathsf{h}_q^c(\mathcal{M}). \tag{1.4.8}$$

Indeed, by Lemma 6.4 of [43], $\mathsf{h}_p^c(\mathcal{M})$ is one-complemented in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$, for $1 \leq p < \infty$. On the other hand, for $1 < p < \infty$ the space $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ is complemented in $L_p(\mathcal{M};\ell_2^c(\mathcal{N}^2))$ via Stein's projection (Theorem 2.13 of [35]), and the column space $L_p(\mathcal{M};\ell_2^c(\mathcal{N}^2))$ is a one-complemented subspace of $L_p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathcal{N}^2)))$. Thus, we conclude from (1.4.2) that, by complementation, (1.4.8) holds.

We turn to the proof of (1.4.6). Step 1 shows that (1.4.6) holds in the case $1 < p \leq 2$. Thus it remains to deal with the case $2 < p < \infty$. We divide the proof in two cases.
Case 1: $1 < q < 2 < p < \infty$. Let $q < s < 2$. Note that $1 < q < s < p$, so there exist $0 < \theta < 1$ and $0 < \phi < 1$ such that $\frac{1-\theta}{1} + \frac{\theta}{s} = \frac{1}{q}$ and $\frac{1-\phi}{q} + \frac{\phi}{p} = \frac{1}{s}$. By (1.4.8) we have

$$\mathsf{h}_s^c(\mathcal{M}) = (\mathsf{h}_q^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\phi.$$

Furthermore, recall that $1 < q < s < 2$, so Step 1 yields

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_\theta.$$

By Wolff's interpolation theorem (1.4.5), it follows that

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\varsigma,$$

where $\varsigma = \frac{\theta\phi}{1-\theta+\theta\phi}$. A simple computation shows that $\frac{1-\varsigma}{1} + \frac{\varsigma}{p} = \frac{1}{q}$.
Case 2: $2 < q < p < \infty$. By a similar argument, we easily deduce this case from the previous one and (1.4.8) using Wolff's theorem.

Note that in both cases, the density assumption of Wolff's theorem is ensured by Remark 1.4.2. □

Lemma 1.4.4. *Let $1 < q < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_{\frac{q}{p}} = \mathsf{h}_p^c(\mathcal{M}). \tag{1.4.9}$$

Proof. Applying the duality theorem 4.5.1 of [8] to (1.4.6) we obtain (1.4.9) in the case $1 < q < p < \infty$ with $\theta = \frac{q}{p}$. Here we used the description of the dual space of $\mathsf{h}_p^c(\mathcal{M})$ for $1 \leq p < \infty$ mentioned in Remark 1.4.2. □

Proof of Theorem 1.4.1. We want to extend (1.4.9) to the case $q = 1$. To this aim we again use Wolff's interpolation theorem combined with the two previous lemmas. Let $1 < q < p < \infty$. Then there exists $0 < \phi < 1$ such that $\frac{1-\phi}{1} + \frac{\phi}{p} = \frac{1}{q}$. We set $\theta = \frac{q}{p}$. Thus by Lemma 1.4.4 we have

$$\mathsf{h}_p^c(\mathcal{M}) = (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_\theta.$$

Moreover we deduce from Lemma 1.4.3 that

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\phi.$$

So Wolff's result yields

$$\mathsf{h}_p^c(\mathcal{M}) = (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_\varsigma,$$

where $\varsigma = \frac{\theta\phi}{1-\theta+\theta\phi}$. An easy computation gives $\varsigma = \frac{1}{p}$, and this ends the proof of (1.4.4) □

The previous results concern the conditioned column Hardy space. We now consider the whole conditioned Hardy space, and get the analogue result.

Theorem 1.4.5. *Let $1 < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p(\mathcal{M}).$$

The proof of Theorem 1.4.5 is similar to that of Theorem 1.4.1. Indeed, we need the analogue of Lemma 1.4.3 for $\mathsf{h}_p(\mathcal{M})$, and the result will follow from the same arguments. By Wolff's result, it thus remains to show that $(\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta = \mathsf{h}_q(\mathcal{M})$ for $1 < p \leq 2$, where $\frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{q}$. Recall that for $1 \leq p \leq 2$ the space $\mathsf{h}_p(\mathcal{M})$ is defined as a sum of three components

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}).$$

We will consider each component, and then will sum the interpolation results. The following lemma describe the behaviour of complex interpolation with addition.

Lemma 1.4.6. *Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces. Then for $0 < \theta < 1$ we have*

$$(A_0, A_1)_\theta + (B_0, B_1)_\theta \subset (A_0 + B_0, A_1 + B_1)_\theta.$$

This result comes directly from the definition of complex interpolation.

Lemma 1.4.7. *Let $1 \leq p_0 < p_1 \leq \infty, 0 < \theta < 1$. Then, the following holds with equivalent norms*

$$(\mathsf{h}_{p_0}^d(\mathcal{M}), \mathsf{h}_{p_1}^d(\mathcal{M}))_\theta = \mathsf{h}_p^d(\mathcal{M})$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Recall that $\mathsf{h}_p^d(\mathcal{M})$ consists of martingale difference sequences in $\ell_p(L_p(\mathcal{M}))$. So $\mathsf{h}_p^d(\mathcal{M})$ is 2-complemented in $\ell_p(L_p(\mathcal{M}))$ for $1 \leq p \leq \infty$ via the projection

$$P : \left\{ \begin{array}{ccc} \ell_p(L_p(\mathcal{M})) & \longrightarrow & \mathsf{h}_p^d(\mathcal{M}) \\ (a_n)_{n\geq 1} & \longmapsto & (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n\geq 1} \end{array} \right. .$$

The fact that $\ell_p(L_p(\mathcal{M}))$ form an interpolation scale with respect to the complex interpolation yields the required result. □

Proof of Theorem 1.4.5 The row version of Lemma 1.4.3 holds true, as well, by considering the equivalent quasinorm N_p^r of $\|\cdot\|_{\mathsf{h}_p^r}$. The diagonal version is ensured by Lemma 1.4.7. Thus Lemma 1.4.6 yields the nontrivial inclusion $\mathsf{h}_q(\mathcal{M}) \subset (\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta$ for $1 < p \leq 2$. On the other hand, by (1.1.1) we have $\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M})$ for $1 < p < \infty$ and (1.2.1) yields by duality the inclusion $\mathsf{h}_1(\mathcal{M}) \subset L_1(\mathcal{M})$. Hence (1.4.2) gives the reverse inclusion $(\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta \subset \mathsf{h}_q(\mathcal{M})$ for $1 < p < \infty$. That establishs the analogue of Lemma 1.4.3 for $\mathsf{h}_p(\mathcal{M})$, and Theorem 1.4.5 follows using duality and Wolff's interpolation theorem. □

We now consider the real method of interpolation. We show that the main result of this section remains true for this method. For $1 < p < \infty$ and $1 \leq r \leq \infty$, similarly to the construction of the space $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ in Remark 1.4.2 we define the column and row subspaces of $L_{p,r}(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathcal{N}^2)))$, denoted by $L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ and $L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^r)$, respectively. Let $\mathsf{h}_{p,r}^c(\mathcal{M})$ be the space of martingales x such that $dx \in L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$.

Theorem 1.4.8. *Let $1 < p < \infty$ and $1 \leq r \leq \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = \mathsf{h}_{p,r}^c(\mathcal{M}). \tag{1.4.10}$$

This result is a corollary of Theorem 1.4.1.

Proof. By a discussion similar to that at the beginning of Step 2 in the proof of Lemma 1.4.3, using (1.4.3) we can show that for $1 < v, s, q < \infty$, $1 \leq r \leq \infty$ and $0 < \eta < 1$ such that $\frac{1}{q} = \frac{1-\eta}{v} + \frac{\eta}{s}$, we have with equivalent norms

$$(\mathsf{h}_v^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_{\eta,r} = \mathsf{h}_{q,r}^c(\mathcal{M}). \tag{1.4.11}$$

We deduce (1.4.10) from (1.4.4) using the reiteration theorem on real and complex interpolations. Let $1 < p < \infty$. Consider $1 < p_0 < p < p_1 < \infty$. There exists $0 < \eta < 1$ such that

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

By Theorem 4.7.2 of [8] we obtain

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = ((\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p_0}}, (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p_1}})_{\eta,r}.$$

Then (1.4.4) yields

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = (\mathsf{h}_{p_0}^c(\mathcal{M}), \mathsf{h}_{p_1}^c(\mathcal{M}))_{\eta,r}.$$

An application of (1.4.11) gives

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = \mathsf{h}_{p,r}^c(\mathcal{M}).$$

This ends the proof of (1.4.10). □

Remark 1.4.9. Musat's result is a corollary of Theorem 1.4.1. By Davis' decomposition proved in [61] we have $\mathcal{H}_p^c(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M})$ for $1 \leq p < 2$. So we can show the analogue of (1.4.6) for $1 < p < 2$ as follows, for $0 < \theta < 1$ and $\frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{q}$

$$\begin{array}{lll} & \mathcal{H}_q^c(\mathcal{M}) & \\ = & \mathsf{h}_q^c(\mathcal{M}) + \mathsf{h}_q^d(\mathcal{M}) & \\ = & (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta + (\mathsf{h}_1^d(\mathcal{M}), \mathsf{h}_p^d(\mathcal{M}))_\theta & \text{by Lemmas 1.4.3 and 1.4.7} \\ \subset & (\mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M}))_\theta & \text{by Lemma 1.4.6} \\ = & (\mathcal{H}_1^c(\mathcal{M}), \mathcal{H}_p^c(\mathcal{M}))_\theta. & \end{array}$$

On the other hand, recall that for $1 \leq p < \infty$, $\mathcal{H}_p^c(\mathcal{M})$ can be identified with the space of all L_p-martingales x such that $dx \in L_p(\mathcal{M}; \ell_2^c)$. Thus we can consider $\mathcal{H}_p^c(\mathcal{M})$ as a subspace of $L_p(\mathcal{M}\overline{\otimes}B(\ell_2))$ and the reverse inclusion follows. Then the same arguments, using duality and Wolff's theorem, yield Theorem 3.1 of [50]. Alternately, we can find Musat's result by defining an equivalent quasinorm for $\|\cdot\|_{\mathcal{H}_p^c(\mathcal{M})}, 0 < p \leq 2$ similar to N_p^c, as follows

$$\tilde{N}_p^c(x) = \inf_W \Big[\tau\Big(\sum_n w_n^{1-2/p}|dx_n|^2\Big)\Big]^{1/2} \approx \|x\|_{\mathcal{H}_p^c(\mathcal{M})}.$$

Then all the previous proofs can be adapted to obtain the analogue results for $\mathcal{H}_p^c(\mathcal{M})$.

Remark 1.4.10. Recall that we define $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) as the Banach space of the $L_\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology. We set $\mathsf{h}_\infty(\mathcal{M}) = \mathsf{h}_\infty^c(\mathcal{M}) \cap \mathsf{h}_\infty^r(\mathcal{M}) \cap \mathsf{h}_\infty^d(\mathcal{M})$. At the time of this writing we do not know if the interpolation result (1.4.4) remains true if we replace $\mathsf{bmo}(\mathcal{M})$ by $\mathsf{h}_\infty(\mathcal{M})$.

Chapter 2

Wavelet approach to operator-valued Hardy spaces

Introduction

In this paper, we exploit Meyer's wavelet methods to the study of the operator-valued Hardy spaces. We are motivated by two rapidly developed fields. The firs one is the theory of noncommutative martingales inequalities. This theory had been already initiated in the 1970's. Its modern period of development has begun with Pisier and Xu's seminal paper [69] in which the authors established the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem between H_1 and BMO. Since then many classical results have been successfully transferred to the noncommutative world (see [43], [46], [52], [4]). In particular, motivated by [37], Mei [52] developed the theory of Hardy spaces on $\mathbb{R}^n$ for operator-valued functions.

Our second motivation is the theory of wavelets founded by Meyer. It is nowadays well known that this theory is important for many domains, in particular in harmonic analysis. For instance, it provides powerful tools to the theory of Calderón-Zygmund singular integral operators. More recently, Meyer's wavelet methods were extended to study more sophistical subjects in harmonic analysis. For example, the authors of [23] exploited the properties of Meyer's wavelets to give a characterization of product BMO by commutators; [57] deals with the estimates of bi-parameter paraproducts.

It is in this spirit that we wish to understand how useful wavelet methods are for noncommutative analysis. The most natural and possible way would be first to do this in the semi-commutative case. This is exactly the purpose of the present paper which could be viewed as the first attempt towards the development of wavelet techniques for noncommutative analysis.

A wavelet basis of $L_2(\mathbb{R})$ is a complete orthonormal system $(w_I)_{I\in\mathcal{D}}$, where $\mathcal{D}$ denotes the collection of all dyadic intervals in $\mathbb{R}$, w is a Schwartz function satisfying the properties needed for Meryer's construction in [49], and

$$w_I(x) \doteq \frac{1}{|I|^{\frac{1}{2}}} w(\frac{x-c_I}{|I|}),$$

where c_I is the center of I. The central facts that we will need about the wavelet basis are the orthogonality between different w_I's, $\|w\|_{L_2(\mathbb{R})} = 1$ and the regularity of w,

$$\max(|w(x)|, |w'(x)|) \lesssim (1+|x|)^{-m}, \quad \forall m \geq 2.$$

The analogy between wavelets and dyadic martingales is well known. The key observation is the following parallelism:

$$\sum_{|I|=2^{-n+1}} \langle f, w_I \rangle w_I \sim df_n,$$

where df_n denotes n-th dyadic martingale difference of f. As dyadic martingales are much easier to handle, this parallelism explains why wavelet approach to many problems in harmonic analysis is usually simple and efficient. On the other hand, it also indicates that martingale methods may be used to deal with wavelets. With this in mind, we develop the operator-valued Hardy spaces based on the wavelet methods in the way which is well known in the noncommutative martingales case. Then we show that our Hardy and BMO spaces coincide with Mei's. In other words, we provide another approach, which is much simpler than Mei's original one, to recover all the results of [52].

This paper is organized as follows. In section 1, we will give some preliminaries on noncommutative analysis, the definition of $\mathcal{H}_p(\mathbb{R},\mathcal{M})$ with $1 \leq p < \infty$ and $L_q\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $2 < q \leq \infty$ in our setting. In section 2, we are concerned with three duality results. The most important one is the noncommutative analogue of the famous Fefferman duality theorem between $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$ and $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. The second one is the duality between $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ and $L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $1 < p < 2$, where we need the noncommutative Hardy-Littlewood maximal inequality, this is why we consider the case $1 < p < 2$ independently. The last one is the duality between $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ and $\mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M})$ with $1 < p < \infty$. As a corollary of the last two results, we identify $\mathcal{H}_q^c(\mathbb{R},\mathcal{M})$ and $L_q^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $2 < q < \infty$. Section 3 deals with the interpolation of our Hardy spaces. In the last section, we show that our Hardy spaces coincide with those of [52]. So, we can give an explicit completely unconditional basis for the space $H_1(\mathbb{R})$, when $H_1(\mathbb{R})$ is equipped with an appropriate operator space structure.

We end this introduction by the convention that throughout the paper the letter c will denote an absolute positive constant, which may vary from lines to lines, and c_p a positive constant depending only on p.

2.1 Preliminaries

2.1.1 Operator-valued noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ and $S_{\mathcal{M}}^+$ be the set of all positive element x in $\mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ is the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. Then any $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$.

Let $1 \leq p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S_{\mathcal{M}}^+$ ($|x| = (x^*x)^{\frac{1}{2}}$). We define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}.$$

One can check that $\|\cdot\|_p$ is well defined and is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$ which is the usual noncommutative L_p- space associated with $(\mathcal{M},\tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements of $L_p(\mathcal{M},\tau)$ can be described as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts). We refer the reader to [70] for more information on noncommutative L_p-spaces.

In this paper, we are concerned with three operator-valued noncommutative L_p-spaces. The first one is the Hilbert-valued noncommutative space $L_p(\mathcal{M}; H^c)$ (resp. $L_p(\mathcal{M}; H^r)$), which is studied at length in [37]. For this space, we need the following properties. In the sequel, p' will always denote the conjugate index of p.

Lemma 2.1.1. *Let $1 \leq p < \infty$. Then*

$$(L_p(\mathcal{M}; H^c))^* = L_{p'}(\mathcal{M}; H^c). \tag{2.1.1}$$

Thus, for $f \in L_p(\mathcal{M}; H^c)$ and $g \in L_{p'}(\mathcal{M}; H^c)$, we have

$$|\tau(\langle f, g\rangle)| \leq \|f\|_{L_p(\mathcal{M};H^c)} \|g\|_{L_{p'}(\mathcal{M};H^c)},$$

where $\langle , \rangle$ denotes the inner product of H.

Lemma 2.1.2. *Let $1 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$[L_{p_0}(\mathcal{M}; H^c), L_{p_1}(\mathcal{M}; H^c)]_\theta = L_p(\mathcal{M}; H^c). \tag{2.1.2}$$

A same equality holds for row spaces.

The second one is the ℓ_∞-valued noncommutative space $L_p(\mathcal{M}; \ell_\infty)$, which is studied by Pisier [65] for an injective $\mathcal{M}$ and Junge [35] for a general $\mathcal{M}$ (see also [43] and [45] for more properties). About this one, we need the following property:

Lemma 2.1.3. *Let $1 \leq p < \infty$. Then*

$$(L_p(\mathcal{M}; \ell_1))^* = L_{p'}(\mathcal{M}; \ell_\infty).$$

Thus, for $x = (x_n)_n \in L_p(\mathcal{M}; \ell_1)$ and $y = (y_n)_n \in L_{p'}(\mathcal{M}; \ell_\infty)$, we have

$$|\sum_{n \geq 1} \tau(x_n y_n)| \leq \|x\|_{L_p(\mathcal{M};\ell_1)} \|y\|_{L_{p'}(\mathcal{M};\ell_\infty)}. \tag{2.1.3}$$

The third one is $L_p(\mathcal{M}; \ell_\infty^c)$ for $2 \leq p \leq \infty$, which was introduced in [18] and is related with the second one by

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)} = \|(|x_n|^2)_n\|_{L_{\frac{p}{2}}(\mathcal{M};\ell_\infty)}.$$

And these are normed spaces by the following characterization

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)} = \inf_{x_n = y_n a} \|(y_n)\|_{\ell_\infty(L_\infty(\mathcal{M}))} \|a\|_{L_p(\mathcal{M})}.$$

We need the interpolation results about these spaces (see [50]):

Lemma 2.1.4. *Let $2 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$[L_{p_0}(\mathcal{M}; \ell_\infty^c), L_{p_1}(\mathcal{M}; \ell_\infty^c)]_\theta = L_p(\mathcal{M}; \ell_\infty^c). \tag{2.1.4}$$

2.1.2 Operator-valued Hardy spaces

In this paper, for simplicity, we denote $L_\infty(\mathbb{R}) \bar{\otimes} \mathcal{M}$ by $\mathcal{N}$. As indicated in the introduction, one can observe that we have the following operator-valued Calderón identity

$$f(x) = \sum_{I \in \mathcal{D}} \langle f, w_I \rangle w_I(x), \tag{2.1.5}$$

which holds when $f \in L_2(\mathcal{N})$. As in the classical case, for $f \in S_{\mathcal{N}}$, we define the two Littlewood-Paley square functions as

$$S_c(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{2.1.6}$$

$$S_r(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f^*, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{2.1.7}$$

For $1 \leq p < \infty$, define

$$\|f\|_{\mathcal{H}_p^c} = \|S_c(f)\|_{L_p(\mathcal{N})},$$
$$\|f\|_{\mathcal{H}_p^r} = \|S_r(f)\|_{L_p(\mathcal{N})}.$$

These are norms, which can be seen easily from the space $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$. So we define the spaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{R},\mathcal{M})$) as the completion of $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$. Now, we define the operator-valued Hardy spaces as follows: for $1 \leq p < 2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) + \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{2.1.8}$$

with the norm

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}_p^c} + \|h\|_{\mathcal{H}_p^r} : f = g + h, g \in \mathcal{H}_p^c, h \in \mathcal{H}_p^r\}$$

and for $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{2.1.9}$$

with the norm defined as

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\}.$$

We can identify $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ as a subspace of $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$, which is related with the two maps below.

Lemma 2.1.5. (i) *The embedding map Φ is defined from $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ to $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$ by*

$$\Phi(f) = \Big(\frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_I\Big)_{I\in\mathcal{D}}. \tag{2.1.10}$$

(ii) *The projection map Ψ is defined from $L_2(\mathcal{N};\ell_2^c(\mathcal{D}))$ to $\mathcal{H}_2^c(\mathbb{R},\mathcal{M})$ by*

$$\Psi((g_I)) = \sum_{I\in\mathcal{D}} \int \frac{g_I}{|I|^{\frac{1}{2}}}\mathbb{1}_I dy \cdot w_I. \tag{2.1.11}$$

2.1.3 Operator-valued $\mathcal{BMO}$ spaces

For $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{\mathcal{BMO}^c} = \sup_{J\in\mathcal{D}} \Big\|\big(\frac{1}{|J|}\sum_{I\subset J} |\langle\varphi, w_I\rangle|^2\big)^{\frac{1}{2}}\Big\|_{\mathcal{M}} \tag{2.1.12}$$

and

$$\|\varphi\|_{\mathcal{BMO}^r} = \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

Define

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\}$$

and

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^r} < \infty\}.$$

These are Banach spaces modulo constant functions. Now we define

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

As in the martingale case [43], we can also define $L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ for all $2 < p \leq \infty$. For $\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{L_p^c\mathcal{MO}} = \Big\|\big(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2\big)_k\Big\|_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} \tag{2.1.13}$$

and

$$\|\varphi\|_{L_p^r\mathcal{MO}} = \|\varphi^*\|_{L_p^c\mathcal{MO}},$$

where I_k^x denote the unique dyadic interval with length 2^{-k+1} that containing x. We will use the convention adopted in [45] for the norm in $L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)$. Thus

$$\Big\|\big(\frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2\big)_k\Big\|_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} = \Big\|{\sup_k}^+ \frac{1}{|I_k^x|}\sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2\Big\|_{L_{\frac{p}{2}}(\mathcal{N})}.$$

Again, we can define

$$L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^c\mathcal{MO}} < \infty\}$$

and

$$L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_r^c\mathcal{MO}} < \infty\}.$$

Define

$$L_p\mathcal{MO}(\mathbb{R},\mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \cap L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}).$$

Note that $L_\infty^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. It easy to check all the spaces we defined here respect to the relevant norms are Banach spaces.

2.2 Duality

To prove the first two duality results in this section, we need the following noncommutative Doob inequality from [35].

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of $\mathcal{N}$.

Lemma 2.2.1. *Let $1 < p \leq \infty$ and $f \in L_p(\mathcal{N})$. Then*

$$\|{\sup_n}^+ \mathcal{E}_n(f)\|_{L_p(\mathcal{N})} \leq c_p \|f\|_{L_p(\mathcal{N})}. \tag{2.2.1}$$

Theorem 2.2.2. *We have*

$$(\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \tag{2.2.2}$$

with equivalent norms. That is, every $\varphi \in \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$ by

$$l_\varphi(f) = \tau \int \varphi^* f, \quad \forall f \in S_\mathcal{N}. \tag{2.2.3}$$

Conversely, for every $l \in (\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^*$, there exits a $\varphi \in \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$ such that $l = l_\varphi$. Moreover,

$$c^{-1}\|\varphi\|_{\mathcal{BMO}^c} \leq \|l_\varphi\|_{(\mathcal{H}_1^c)^*} \leq c\|\varphi\|_{\mathcal{BMO}^c}$$

where $c > 0$ is a universal constant.

Similarly, the duality holds between $\mathcal{H}_1^r$ and $\mathcal{BMO}^r$, between $\mathcal{H}_1$ and $\mathcal{BMO}$ with equivalent norms.

In order to adapt the arguments in the martingale case, we need to define the truncated square functions for $n \in \mathbb{Z}$,

$$S_{c,n}(f)(x) = \Big(\sum_{k=-\infty}^{n}\sum_{|I|=2^{-k+1}}\frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}.$$

Proof. Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$, by an approximation argument, we only need to prove the inequality

$$|l_\varphi(f)| \leq c\|\varphi\|_{\mathcal{BMO}^c}\|f\|_{\mathcal{H}_1^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in $\mathcal{M}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then we have

$$\begin{aligned}
|l_\varphi(f)| &= |\tau\int\varphi^* f dx| \\
&= \Big|\sum_n \tau\int\sum_{|I|=2^{-n+1}}\langle\varphi, w_I\rangle^* w_I\sum_{|I'|=2^{-n+1}}\langle f, w_{I'}\rangle w_{I'}dx\Big| \\
&= \Big|\sum_n \tau\int\sum_{|I|=2^{-n+1}}\frac{\langle\varphi, w_I\rangle^*}{|I|^{\frac{1}{2}}}\mathbb{1}_I\sum_{|I'|=2^{-n+1}}\frac{\langle f, w_{I'}\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_{I'}dx\Big| \\
&\leq \sum_n\Big(\tau\int|\sum_{|I|=2^{-n+1}}\frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_I|^2 S_{c,n}^{-1}(f)\Big)^{\frac{1}{2}} \\
&\quad\cdot\Big(\tau\int|\sum_{|I|=2^{-n+1}}\frac{\langle\varphi, w_I\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_I|^2 S_{c,n}(f)\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_n\tau\int\sum_{|I|=2^{-n+1}}\frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I S_{c,n}^{-1}(f)\Big)^{\frac{1}{2}} \\
&\quad\cdot\Big(\sum_n\tau\int\sum_{|I|=2^{-n+1}}\frac{|\langle\varphi, w_I\rangle|^2}{|I|}\mathbb{1}_I S_{c,n}(f)\Big)^{\frac{1}{2}} \\
&= A\cdot B.
\end{aligned}$$

In the above estimates, the first equality has used the orthogonality of the w_I's on different levels, the second the orthogonality of the w_I's on the same level and the disjoint of different dyadic I's on the same level; the first inequality has used the Hölder inequality in Lemma 2.1.1, and the second the Cauchy-Schwarz inequality and the disjointness of different I's on the same level.

Now, let us estimate A:

$$\begin{aligned}
A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f))S_{c,n}^{-1}(f) \\
&= \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f))(1 + S_{c,n-1}(f)S_{c,n}^{-1}(f)) \\
&\le \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f))\|1 + S_{c,n-1}(f)S_{c,n}^{-1}(f)\|_\infty \\
&\le 2\sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f)) \\
&= 2\|f\|_{\mathcal{H}_1^c}.
\end{aligned}$$

For the first inequality, we has used the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$.

The second term is estimated as follows:

$$\begin{aligned}
B^2 &= \sum_k \tau \int (S_{c,k}(f) - S_{c,k-1}(f)) \sum_{n\ge k} \sum_{|I|=2^{-n+1}} \frac{|\langle\varphi, w_I\rangle|^2}{|I|}\mathbb{1}_I \\
&= \sum_k \tau \sum_j (S_{c,k}(f) - S_{c,k-1}(f)) \int_{I_k^j} \sum_{n\ge k} \sum_{|I|=2^{-n+1}} \frac{|\langle\varphi, w_I\rangle|^2}{|I|}\mathbb{1}_I \\
&= \sum_k \tau \sum_j \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \frac{1}{|I_k^j|} \sum_{I\subset I_k^j} |\langle\varphi, w_I\rangle|^2 \\
&\le \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \Big\| \frac{1}{|I_k^j|} \sum_{I\subset I_k^j} |\langle\varphi, w_I\rangle|^2 \Big\|_\infty \\
&\le \|\varphi\|_{\mathcal{BMO}^c}^2 \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \\
&= \|\varphi\|_{\mathcal{BMO}^c}^2 \|f\|_{\mathcal{H}_1^c}
\end{aligned}$$

The fist equality has used the Fubini theorem, the second one the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic interval $I_k^j = [j2^{-k+1}, (j+1)2^{-k+1})$; the first inequality has used the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$.

Now, let us begin to deal with another direction, i.e. suppose that l is a bounded linear functional on $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, we want to find an operator-valued function φ in $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$, such that $l = l_\varphi$ and $l_\varphi(f) = \tau \int \varphi^* f$ for $f \in S_\mathcal{N}$. By the embedding operator Φ in (2.1.10) and by the Banach-Hahn theorem, l extends to a bounded continuous functional on $L_1(\mathcal{N}; \ell_2^c(\mathcal{D}))$ of the same norm. Then by the results in Lemma 2.1.1 there exists $g = (g_I)_{I\in\mathcal{D}}$ such that $\|g\|_{L_\infty(\mathcal{N};\ell_2^c(\mathcal{D}))} = \|l\|$, and

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_I, \quad \forall f \in S_\mathcal{N}.$$

Now let $\varphi = \Psi(g)$, where Ψ is defined as (2.1.11). The orthogonality of the w_I's yields

$$\begin{aligned}
\|\sum_{I\subset J} |\langle\varphi, w_I\rangle|^2\|_\mathcal{M} &= \|\sum_{I\subset J} |\int \frac{g_I}{|I|^{\frac{1}{2}}}\mathbb{1}_I|^2\|_\mathcal{M} \le \|\sum_{I\subset J} \int_J |g_I|^2\|_\mathcal{M} \\
&\le |J| \|\sum_{I\subset J} |g_I|^2\|_{L_\infty(\mathcal{N})} \le |J| \|(g_I)_I\|_{L_\infty(\mathcal{N};\ell_2^c(\mathcal{D}))},
\end{aligned}$$

where the first inequality used the Kadison-Schwartz inequality. Also thanks to the orthogonality of the w_I's, we get

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f$$

for all $f \in S_{\mathcal{N}}$. Therefore, we complete the proof about $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$ and $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. Passing to adjoint, we have the conclusion concerning $\mathcal{H}_1^r$ and $\mathcal{BMO}^r$. Finally, by the classical fact that the dual of a sum space is the intersection space, we obtain the duality between $\mathcal{H}_1$ and $\mathcal{BMO}$. □

Theorem 2.2.3. *Let $1 < p < 2$. We have*

$$(\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^* = L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \tag{2.2.4}$$

with equivalent norms. That is, every $\varphi \in L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ by

$$l_\varphi(f) = \tau \int \varphi^* f, \quad \forall f \in S_{\mathcal{N}}. \tag{2.2.5}$$

Conversely, for every $l \in (\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^*$, there exists an operator-valued function $\varphi \in L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ such that $l = l_\varphi$ and

$$c_p^{-1}\|\varphi\|_{L_{p'}^c\mathcal{MO}} \leq \|l_\varphi\|_{(\mathcal{H}_p^c)^*} \leq \sqrt{2}\|\varphi\|_{L_{p'}^c\mathcal{MO}}$$

Similarly, the duality holds between $\mathcal{H}_p^r$ and $L_{p'}^r$, between $\mathcal{H}_p$ and $L_{p'}\mathcal{MO}$ with equivalent norms.

We need the following lemma of [43]. We write it down for convenience of the reader but without proof.

Lemma 2.2.4. *Let s, t be two real numbers such that $s < t$ and $0 \leq s \leq 1 \leq t \leq 2$. Let x, y be two positive operators such that $x \leq y$ and $x^{t-s}, y^{t-s} \in L_1(\mathcal{N})$. Then*

$$\tau \int y^{-s/2}(y^t - x^t)y^{-s/2} \leq 2\tau \int y^{-(s+1-t)/2}(y-x)y^{-(s+1-t)/2}.$$

Proof. We need only to prove the first assertion on $\mathcal{H}_p^c$. Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$, by an approximation argument, we only need to prove the inequality

$$|l_\varphi(f)| \leq c\|\varphi\|_{L_{p'}^c\mathcal{MO}}\|f\|_{\mathcal{H}_p^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in $\mathcal{M}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. By the similar principle in the noncommutative martingale case as in

[43], we have

$$
\begin{aligned}
|l_\varphi(f)| &= |\tau \int \varphi^* f dx| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I\rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'}\rangle w_{I'} dx\Big| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I\rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \sum_{|I'|=2^{-n+1}} \frac{\langle f, w_{I'}\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_{I'} dx\Big| \\
&\le \sum_n \Big(\tau \int |\sum_{|I|=2^{-n+1}} \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I|^2 S_{c,n}^{p-2}(f)\Big)^{\frac{1}{2}} \\
&\quad \cdot \Big(\tau \int |\sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I|^2 S_{c,n}^{2-p}(f)\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{p-2}(f)\Big)^{\frac{1}{2}} \\
&\quad \cdot \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{2-p}(f)\Big)^{\frac{1}{2}} \\
&= A \cdot B.
\end{aligned}
$$

Now we need the above lemma to estimate the first term. Take $s = 2 - p$ and $t = 2$, the lemma yields

$$
\begin{aligned}
A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{p-2}(f) \\
&= \sum_n \tau \int S_{c,n}^{-(2-p)/2}(f)(S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{-(2-p)/2}(f) \\
&\le 2 \sum_n \tau \int S_{c,n}^{-(1-p)/2}(f)(S_{c,n}(f) - S_{c,n-1}(f)) S_{c,n}^{-(1-p)/2}(f) \qquad (2.2.6) \\
&= 2 \sum_n \tau \int S_{c,n}(f) - S_{c,n-1}(f) S_{c,n}^{p-1}(f) \\
&\le 2 \sum_n \tau \int S_{c,n}^p(f) - S_{c,n-1}^p(f) = 2\|f\|_{\mathcal{H}_p^c}^p.
\end{aligned}
$$

The last inequality used two elementary inequalities: $0 \le S_{c,n-1}(f) \le S_{c,n}(f)$ implies $S_{c,n-1}^{p-1}(f) \le S_{c,n}^{p-1}(f)$ for $0 < p - 1 < 1$; and $\tau(S_{c,n-1}^{p-1}(f)) \le \tau(S_{c,n-1}^{\frac{1}{2}}(f) S_{c,n}^{p-1}(f) S_{c,n-1}^{\frac{1}{2}}(f))$.

The second term can be deduced from the nontrivial duality results in Lemma 2.1.3

for $1 < p < \infty$ as follows.

$$\begin{aligned}
B^2 &= \sum_k \tau \int S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \sum_{n\geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \int_{I_k^j} \sum_{n\geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j \int \mathbb{1}_{I_k^j}(x) S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^j|} \sum_{I\subset I_k^j} |\langle \varphi, w_I\rangle|^2 dx \\
&= \sum_k \tau \int S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2 dx \\
&\leq \|\sum_k S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f)\|_{L_{(p'/2)'}} \Big\| \sup_k \frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2 \Big\|_{L_{p'/2}} \\
&= \|\varphi\|^2_{L^c_{p'}\mathcal{MO}} \|f\|^{2-p}_{\mathcal{H}_p^c}.
\end{aligned}$$

The fist equality has used the Fubini theorem, the second one the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic intervals with length 2^{-k+1}.

For another direction, we can carry out the proof as that in the case $p = 1$. Suppose that l is a bounded linear functional on $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$. By the embedding operator Φ and by Hahn-Banach theorem, and the results in Lemma 2.1.1

we can find $g = (g_I)_{I\in\mathcal{D}}$ such that $\|g\|_{L_{p'}(\mathcal{N};\ell_2^c(\mathcal{D}))} = \|l\|$ and

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I, \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$ defined in (2.1.11), the orthogonality of the w_I's yields

$$\begin{aligned}
&\|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} |\langle \varphi, w_I\rangle|^2\|_{L_{q/2}(\mathcal{N})} \\
&= \|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} |\int \frac{g_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I|^2\|_{L_{q/2}(\mathcal{N})} \\
&\leq \|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} \int_{I_n^x} |g_I|^2\|_{L_{q/2}(\mathcal{N})} \\
&\leq \|\sup_n{}^+ \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I\subset I_n^x} |g_I|^2\|_{L_{q/2}(\mathcal{N})} \\
&\leq \|\sup_n{}^+ \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I\in\mathcal{D}} |g_I|^2\|_{L_{q/2}(\mathcal{N})} \\
&\leq c\| \sum_{I\in\mathcal{D}} |g_I|^2\|_{L_{q/2}(\mathcal{N})} \\
&= c\|(g_I)_I\|_{L_{q/2}(\mathcal{N};\ell_2^c(\mathcal{D}))},
\end{aligned}$$

where for the first inequality we used the Kadison-Schwartz inequality, and the last inequality is (2.2.1). Also due to the orthogonality of the w_I's, we get

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f,$$

for all $f \in S_{\mathcal{N}}$. Therefore, we complete the proof about $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ and $L^c_{p'}\mathcal{MO}(\mathbb{R}, \mathcal{M})$. $\square$

Instead of using the noncommutative Doob inequality, we will use the following noncommutative Stein inequality from [69] to prove the duality between the spaces $\mathcal{H}_p^c$, $1 < p < \infty$.

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of $\mathcal{N}$.

Lemma 2.2.5. *Let $1 < p < \infty$ and $a = (a_n)_n \in L_p(\mathcal{N}; \ell_2^c)$. Then there exists a constant depending only on p such that*

$$\Big\|\big(\sum_n |\mathcal{E}_n a_n|^2\big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{N})} \leq C_p \Big\|\big(\sum_n |a_n|^2\big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{N})}. \tag{2.2.7}$$

Theorem 2.2.6. *For any $1 < p < \infty$, we have*

$$(\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{R}, \mathcal{M}), \tag{2.2.8}$$

Proof. By a similar reason as in the corresponding part of the proof of Theorem 2.2.2, we can carry out the following calculation,

$$\begin{aligned}
|l_\varphi(f)| &= |\tau \int \varphi^* f dx| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I\rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'}\rangle w_{I'} dx\Big| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I\rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I dx\Big| \\
&\leq \|\big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I\big)^{\frac{1}{2}}\|_{L_p(\mathbb{R},\mathcal{M})} \cdot \|\big(\sum_{I\in\mathcal{D}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I\big)^{\frac{1}{2}}\|_{L_{p'}(\mathbb{R},\mathcal{M})}.
\end{aligned}$$

Now, we turn to the proof of the inverse direction. Take a bounded linear functional $l \in (\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^*$, by the embedding operator Φ and the Hahn-Banach extension theorem, l extends to a bounded linear functional on $L_p(\mathcal{N}; \ell_2^c)$ with the same norm. Thus by (2.1.1), there exists a sequence $g = (g_I)_I$ such that

$$\|g\|_{L_q(\mathcal{N}; l_2^c(\mathcal{D}))} = \|l\|$$

and

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_p^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I, \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$ where Ψ is defined in (2.1.11), then applying the Stein inequality (2.2.5) to the conditional expectation

$$\mathcal{E}_I(h) = \sum_J \frac{1}{|J|} \int_J h(y) dy \cdot \mathbb{1}_J,$$

where J is dyadic interval with the same length as I, we get

$$\begin{aligned}
\|\varphi\|_{\mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M})} &= \|\big(\sum_{I\in\mathcal{D}} |\frac{1}{|I|} \int_I g_I dy \cdot \mathbb{1}_I|^2\big)^{\frac{1}{2}}\|_{L_{p'}(\mathcal{N})} \\
&\leq \|\big(\sum_{I\in\mathcal{D}} |\mathcal{E}_I(g_I)|^2\big)^{\frac{1}{2}}\|_{L_{p'}(\mathcal{N})} \\
&\leq c_{p'} \|\big(\sum_{I\in\mathcal{D}} |g_I|^2\big)^{\frac{1}{2}}\|_{L_{p'}(\mathcal{N})}.
\end{aligned}$$

By the orthogonality of the w_I's, we have

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f,$$

for all $f \in S_{\mathcal{N}}$. $\square$

From the proof of the second part of Theorem 2.2.2, Theorem 2.2.3 and Theorem 2.2.6, we state the boundedness of Ψ as a corollary.

Corollary 2.2.7. (i) *Let $1 < p < \infty$, Ψ is a projection map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ onto $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ if we identify the latter as a subspace of the former.*

(ii) *Let $2 < p \leq \infty$, Ψ is also a bounded map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ to $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$.*

Theorem 2.2.3 and Theorem 2.2.6 immediately imply the following corollary:

Corollary 2.2.8. *Let $2 < p < \infty$. Then*

$$\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M}), \quad \forall 2 < p < \infty$$

with equivalent norms.

However, for the part $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M}) \subset \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$, we can give another proof. The idea is essentially similar to that in [52], the good news is that in our case, the argument seems very elegant. Now we give the detailed proof.

Proof. Our tent space is defined as

$$T_p^c = \Big\{ f = \{f_I\}_I \in L_p(\mathcal{M}; \ell_2^c(\mathcal{D})) : \quad \tau \int \big(\sum_{I\in\mathcal{D}} \frac{f_I^2}{|I|} \mathbb{1}_I \big)^{\frac{p}{2}} < \infty \Big\}.$$

We claim that every $\varphi \in L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$ induces a bounded linear functional on $T_{p'}^c$,

$$l_\varphi(f) = \tau \int \sum_{I\in\mathcal{D}} \frac{\langle \varphi, w_I\rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I dx$$

and $\|l_\varphi\| \leq \|\varphi\|_{L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M})}$. The proof is just the copy of the proof of the first part in the last theorem. Now $T_{p'}^c$ is naturally embedded into $L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D}))$ by $(f_I)_I \to (\frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I)_I$. So by the Hahn-Banach extension theorem, l_φ extends to an bounded linear functional on $L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D}))$ with the same norm. Then by the duality between

$$(L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D})))^* = L_p(\mathcal{N}; \ell_2^c(\mathcal{D})).$$

there exists a unique $h = (h_I)_I$ such that $\|h\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))} \leq \|l_\varphi\|$ and for $f = (f_I)_I \in T_{p'}^c$,

$$l_\varphi(f) = \tau \int \sum_{I\in\mathcal{D}} h_I^* \frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I dx.$$

So we get

$$\frac{\langle \varphi, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = h_I,$$

thus,

$$\begin{aligned} \|\varphi\|_{\mathcal{H}_p^c} &= \Big\| \big(\sum_{I\in\mathcal{D}} \frac{\langle \varphi, w_I\rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{N})} \\ &= \|h_I\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))} \leq \|l_\varphi\|. \end{aligned}$$

$\square$

2.3 Interpolation

This section is devoted to the interpolation of our wavelet Hardy spaces. The interpolation results below will be needed in the next section to compare our Hardy spaces with those of Mei.

Lemma 2.3.1. *Let $1 < p_0 < p < p_1 < \infty$, we have*

$$[\mathcal{H}^c_{p_0}(\mathbb{R},\mathcal{M}),\mathcal{H}^c_{p_1}(\mathbb{R},\mathcal{M})]_\theta = \mathcal{H}^c_p(\mathbb{R},\mathcal{M}) \tag{2.3.1}$$

with equivalent norms, where θ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. The embedding map Φ yields

$$[\mathcal{H}^c_{p_0},\mathcal{H}^c_{p_1}]_\theta \subset \mathcal{H}^c_p.$$

On the other hand, it is the boundedness of the projection map Ψ from $L_p(\mathcal{N};\ell^c_2(\mathcal{D}))$ to $\mathcal{H}^c_p(\mathbb{R},\mathcal{M})$ stated in Corollary 2.2.7 that yields the inverse direction. □

Theorem 2.3.2. *Let $1 \le q < p < \infty$, we have*

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}),\mathcal{H}^c_q(\mathbb{R},\mathcal{M})]_{\frac{q}{p}} = \mathcal{H}^c_p(\mathbb{R},\mathcal{M}) \tag{2.3.2}$$

with equivalent norms.

Proof. We will prove the theorem by a general strategy as appeared in [50].

Step 1: We prove the conclusion for $2 < q < p < \infty$:

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}),\mathcal{H}^c_q(\mathbb{R},\mathcal{M})]_{\frac{q}{p}} = \mathcal{H}^c_p(\mathbb{R},\mathcal{M}). \tag{2.3.3}$$

The identity can be seen easily from the following two inclusions. On one hand, the operator Φ which in (2.1.10), together with (2.1.2) yields

$$[\mathcal{H}^c_1(\mathbb{R},\mathcal{M}),\mathcal{H}^c_{q'}(\mathbb{R},\mathcal{M})]_{\frac{q}{p}} \subset \mathcal{H}^c_{p'}(\mathbb{R},\mathcal{M}).$$

Then by duality and Corollary 2.2.8, we have

$$L^c_p\mathcal{MO}(\mathbb{R},\mathcal{M}) \subset [\mathcal{BMO}^c(\mathbb{R},\mathcal{M}),L^c_q\mathcal{MO}(\mathbb{R},\mathcal{M})]_{\frac{q}{p}}. \tag{2.3.4}$$

On the other hand, the operator $\mathcal{T}$ identifying $L^c_p\mathcal{MO}(\mathbb{R},\mathcal{M})$ as a subspace of $L_p(L_\infty(\mathcal{N}\bar{\otimes}B(\ell_2(\mathcal{D}));\ell^c_\infty)$ defined by

$$\mathcal{T}(\varphi) = \langle f, w_I\rangle |I^t_k|^{-\frac{1}{2}} \mathbb{1}_{I\subset I^t_k}(I)\otimes e_{I,1}, \tag{2.3.5}$$

together with Lemma 2.1.4 yields

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}),L^c_q\mathcal{MO}(\mathbb{R},\mathcal{M})]_{\frac{q}{p}} \subset L^c_p\mathcal{MO}(\mathbb{R},\mathcal{M}). \tag{2.3.6}$$

Step 2: we prove the conclusion for $1 < q < p < \infty$. This step can be divided into two substeps.

Substep 21: $p > 2$. Let $p < s < \infty$. By Step 1, we have

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}),\mathcal{H}^c_p(\mathbb{R},\mathcal{M})]_{\frac{p}{s}} = \mathcal{H}^c_s(\mathbb{R},\mathcal{M}).$$

On the other hand, by Theorem 2.3.1, we have

$$[\mathcal{H}_q^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c,$$

where(and in the rest of the paper) θ denote the interpolation parameter. Then Wolff's interpolation theorem yields the result.

Substep 22: $p \leq 2$. Let $s > 2$, then by Substep 21, we have

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})]_{\frac{p}{s}} = \mathcal{H}_s^c(\mathbb{R}, \mathcal{M}).$$

Then together with Lemma 2.3.1, Wolff's interpolation theorem yields the result.

Step 3: we prove the conclusion for $1 = q < p < \infty$. Take $s > \max(p, 2)$. By Step 2 and duality [8, Theorem 4.3.1], we get

$$[\mathcal{H}_1^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c.$$

Then together with Step 2, Wolff's interpolation yields the conclusion. □

Remark 2.3.3. If one can directly prove Lemma 2.3.1 for $p_0 = 1$, we can prove the above theorem without the help of $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$ for $2 < p < \infty$ as carried out in [4], where one needs an auxiliary space.

Theorem 2.3.4. *For $1 < p < \infty$, we have*

$$\mathcal{H}_p(\mathbb{R}, \mathcal{M}) = L_p(\mathcal{N})$$

with equivalent norms.

Proof. There are several ways to prove this result. One can prove it by the strategy in [69] together with Stein's inequality (2.2.5). Here, we just use the fact that $L_p(\mathcal{M})$ with $1 < p < \infty$ is a UMD space and our $(w_I)_I$ is an complete orthonormal basis. So by Theorem 3.8 in [31], we have

$$\|f\|_{L_p(\mathcal{N})} \simeq \Big(\mathbb{E}\Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \Big\|^p_{L_p(\mathcal{N})}\Big)^{\frac{1}{p}}.$$

Then we complete the proof for $2 \leq p < \infty$ by Khintchine's inequalities. Now, let us prove the case $1 < p < 2$. Let $f \in \mathcal{H}_p(\mathbb{R}, \mathcal{M})$, then for any $\epsilon > 0$, by the definition of $\mathcal{H}_p(\mathbb{R}, \mathcal{M})$, there exists a decomposition $f = f_c + f_r$ such that

$$\|f_c\|_{\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})} + \|f_r\|_{\mathcal{H}_p^r(\mathbb{R}, \mathcal{M})} \leq \|f\|_{\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})} + \epsilon.$$

Take any $g \in L_{p'}(\mathcal{N})$, by the results for $p' > 2$, the operator valued Calderón identity (2.1.5) yields

$$\begin{aligned}
|\tau \int g f^*| &= |\sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\leq |\sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_c^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\quad + |\sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_r^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\leq \|S_c(g)\|_{L_{p'}(\mathcal{N})} \|S_c(f_c)\|_{L_p(\mathcal{N})} + |S_r(g)\|_{L_{p'}(\mathcal{N})} \|S_r(f_r)\|_{L_p(\mathcal{N})}
\end{aligned}$$

$$\leq c_{p'}\|g\|_{L_{p'}}(\|f\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})}+\epsilon).$$

Taking sup and let $\epsilon \to 0$, we get the required result.

Finally, we prove the inverse inequality. Let $f \in L_p(\mathcal{N})$, by duality, we can find two sequences of functions $(F_{c,I})_I \in L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$ and $(F_{r,I})_I \in L_p(\mathcal{N};\ell_2^r(\mathcal{D}))$ such that $F_{c,I}+F_{r,I} = \langle f, w_I\rangle |I|^{-\frac{1}{2}}\mathbb{1}_I$ and

$$\|(F_{c,I})_I\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))}+\|(F_{r,I})_I\|_{L_p(\mathcal{N};\ell_2^r(\mathcal{D}))} \leq \|f\|_{L_p(\mathcal{N})}.$$

Let $f_c = \Psi((F_{c,I})_I)$ and $f_r = \Psi((F_{r,I})_I)$, by identity (2.1.5), we have $f = f_c + f_r$. On the other hand, by the Stein inequality (2.2.5), we have $\|f_c\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})} \leq \|(F_{c,I})_I\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))}$ and $\|f_r\|_{\mathcal{H}_p^r(\mathbb{R},\mathcal{M})} \leq \|(F_{r,I})_I\|_{L_p(\mathcal{N};\ell_2^r(\mathcal{D}))}$. So we have found the desired decomposition of f. □

Theorem 2.3.5. *The following results hold with equivalent norms:*

(i) *Let $1 \leq q < p < \infty$, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} = L_p(\mathcal{N}). \tag{2.3.7}$$

(ii) *Let $1 < q < p \leq \infty$, we have*

$$[\mathcal{H}_1(\mathbb{R},\mathcal{M}), L_p(\mathcal{N})]_{\frac{p'}{q'}} = L_q(\mathcal{N}). \tag{2.3.8}$$

(iii) *Let $1 < p < \infty$, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_{\frac{1}{p}} = L_p(\mathcal{N}). \tag{2.3.9}$$

In order to prove this theorem, we need the following result from the theory of interpolation. We formulate it here without proof.

Lemma 2.3.6. *Let A_0, B_0, A_1, B_1 be four Banach spaces satisfying the property needed of interpolation. Then*

$$[A_0+B_0, A_1+B_1]_\theta \supset [A_0,A_1]_\theta + [B_0,B_1]_\theta$$

and

$$[A_0\cap B_0, A_1\cap B_1]_\theta \subset [A_0,A_1]_\theta \cap [B_0,B_1]_\theta.$$

Proof. (i) We also exploit the similar but different strategy with that in the proof of Theorem 2.3.2.

Step 1: we prove the results for $2 \leq q < p < \infty$. By Theorem 2.3.4, Theorem 2.3.2 and the lemma, we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} \subset L_p(\mathcal{N}).$$

The inverse direction follows from $L_\infty(\mathcal{N}) \subset \mathcal{BMO}(\mathbb{R},\mathcal{M})$,

$$\begin{aligned} L_p(\mathcal{N}) &= [L_\infty(\mathcal{N}), L_q(\mathcal{N})]_{\frac{q}{p}} \\ &\subset [\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} \end{aligned}$$

Step 2: we prove the results for $1 \leq q < 2 \leq p < \infty$. By Step 1, we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_2(\mathcal{N})]_{\frac{2}{p}} = L_p(\mathcal{N}).$$

Together with

$$L_2(\mathcal{N}) = [L_p(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff's interpolation yields the conclusion.

Step 3: we prove the results for $1 \le q < p < 2$. By Step 2, we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{2}} = L_2(\mathcal{N}).$$

Together with

$$L_p(\mathcal{N}) = [L_2(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff's interpolation yields the conclusion.

(ii) The results for $1 < q < p < \infty$ can be immediately proved by duality and the partial results in (i). For $p = \infty$, take $q < s < \infty$, then by Wolff's argument, we get the conclusion.

(iii) First, we prove conclusion for $p < 2$. Then by (i) and (ii), we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{p'}} = L_{p'}(\mathcal{N})$$

and

$$[\mathcal{H}_1(\mathbb{R}, \mathcal{M}), L_{p'}(\mathcal{N})]_{\frac{p}{p'}} = L_p(\mathcal{N}).$$

Therefore, we end with Wolff's argument. Second, the proof for $p > 2$ is the same. At last, when $p = 2$, we can take $s > 2$, by the results for $p \neq 2$ and reiteration theorem in [8, Theorem 4.6.1], we get

$$\begin{aligned} L_2 &= [L_s, L_{s'}]_\theta = [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{s}}, \mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{s'}}]_\theta \\ &= [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_\theta. \end{aligned}$$

□

2.4 Comparison with Mei's results

We denote the column Hardy space $H_p^c(\mathbb{R}, \mathcal{M})$ and the bounded mean oscillation space $BMO^c(\mathbb{R}, \mathcal{M})$ in [52]. We have the following result.

Theorem 2.4.1. *We have*

$$\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) = BMO^c(\mathbb{R}, \mathcal{M})$$

with equivalent norms. Similar results holds for the row spaces. Consequently, $\mathcal{BMO}(\mathbb{R}, \mathcal{M}) = BMO(\mathbb{R}, \mathcal{M})$ *with equivalent norms.*

The theorem can be easily seen from the corresponding $BMO(\mathbb{R}, H)$-spaces. However, we can exploit the idea of [31] to prove our $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ also coincide with that defined by the mean oscillation.

Proof. $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) \subset BMO^c(\mathbb{R}, \mathcal{M})$. Let $\varphi \in \mathcal{BMO}_c(\mathbb{R}, \mathcal{M})$. As in [31], fix a finite interval $I \subset \mathbb{R}$, and consider the collections of dyadic intervals

(1) $\mathcal{D}_1 := \{J \in \mathcal{D}; 2|J| > |I|\}$'

(2) $\mathcal{D}_2 := \{J \in \mathcal{D}; 2|J| \le |I|, 2J \cap 2I = \emptyset\}$,

(3) $\mathcal{D}_3 := \{J \in \mathcal{D}; 2|J| \leq |I|, 2J \cap 2I \neq \emptyset\}$.

Let $a_J = \langle \varphi, \omega_J \rangle$, then we have a priori formal series

$$\varphi_1(x) = \sum_{J \in \mathcal{D}_1} a_J[\omega_J(x) - \omega_J(c_I)], \varphi_i(x) = \sum_{J \in \mathcal{D}_i} a_J \omega_J(x), i = 2, 3,$$

where c_I is the center of the interval I. Denote $\varphi_I = \varphi_1 + \varphi_2 + \varphi_3$, by a similar discussion in [31], we only need to prove:

$$\|\frac{1}{|I|} \int_I |\varphi_I(x)|^2 dx\|_{\mathcal{M}} < \infty.$$

By scaling we can assume:

$$\sup_I \frac{1}{|I|} \| \sum_{J \subset I} |a_J|^2 \| = 1.$$

Then we have the following obvious bound for individual terms $\|a_J\| \leq |J|^{\frac{1}{2}}$.

Estimates for φ_1:

$$\begin{aligned}
\|\frac{1}{|I|} \int_I |\varphi_1(x)|^2 dx\| &\leq \frac{1}{|I|} (\sum_{J \in \mathcal{D}_1} \|a_J\| |\omega_J(x) - \omega_J(c_I)|)^2 dx \\
&\leq c \frac{1}{|I|} \int_I [\sum_{J \in \mathcal{D}_1} |J|^{\frac{1}{2}} |I| |J|^{-\frac{3}{2}} (1 + \frac{dist(I, J)}{|J|})^{-2}]^2 dx \\
&= c[\sum_{j=0}^{\infty} \sum_{|J| \in (2^{j-1}, 2^j]|I|} |I| |J|^{-1} (1 + \frac{dist(I, J)}{|J|})^{-2}]^2 < \infty.
\end{aligned}$$

Estimates for φ_2:

$$\begin{aligned}
\|\frac{1}{|I|} \int_I |\varphi_2(x)|^2 dx\| &\leq \frac{1}{|I|} \int_I \| \sum_{\mathcal{D}_2} a_J \omega_J(x) \|^2 dx \\
&\leq \frac{1}{|I|} \int_I (\sum_{\mathcal{D}_2} \|a_J\| |\omega_J(x)|)^2 dx \\
&\leq c \frac{1}{|I|} \int_I [\sum_{\mathcal{D}_2} |J|^{\frac{1}{2}} |J|^{-\frac{1}{2}} (\frac{dist(I, J)}{|J|})^{-2}]^2 dx \\
&= c[\sum_{j=1}^{\infty} \sum_{|J| \in (2^{-j-1}, 2^{-j})|I|, dist(I,J) > 2^{-1}|I|} (\frac{dist(I, J)}{|J|})^{-2}]^2 < \infty.
\end{aligned}$$

Estimates for φ_3:

$$\|\frac{1}{|I|} \int_I |\varphi_3(x)|^2 dx\| \leq \frac{1}{|I|} \| \sum_{J \in \mathcal{D}_3} |a_J|^2 \| \leq \frac{1}{|I|} \| \sum_{J \subset 4I} |a_J|^2 \| < \infty$$

Hence we deduce that:

$$\| \int_I |\varphi_I(x)|^2 dx\|_{\mathcal{M}} \leq c \sum_{i=1}^{3} \| \int_I |\varphi_i(x)|^2 dx\|_{\mathcal{M}} \leq c|I|$$

Now turn to the proof of inverse direction $BMO^c(\mathbb{R}, \mathcal{M}) \subset \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$. Let $\varphi \in BMO^c(\mathbb{R}, \mathcal{M})$. The proof is very similar to that in Mei's work [52]. For any dyadic interval $I \subset \mathbb{R}$, write $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where $\varphi_1 = (\varphi - \varphi_{2I})\chi_{2I}, \varphi_2 = (\varphi - \varphi_{2I})\chi_{2I^c}, \varphi_3 = \varphi_{2I}$.

Thus

$$\sum_{J\subset I} |\langle \varphi, \omega_J\rangle|^2 \leq 2(\sum_{J\subset I} |\langle \varphi_1, \omega_J\rangle|^2 + \sum_{J\subset I} |\langle \varphi_2, \omega_J\rangle|^2)$$

Estimates for φ_1:

$$\| \sum_{J\subset I} |\langle \varphi_1, \omega_J\rangle|^2\| \leq \| \int |\varphi_1(x)|^2 dx\| \leq c\| \int_{2I} |\varphi - \varphi_{2I}|^2\| \leq c|I|$$

Estimates for φ_2:

$$\begin{aligned}
\| \sum_{J\subset I} |\langle \varphi_2, \omega_J\rangle|^2\| &= \| \sum_{J\subset I} |\sum_{k=1}^{\infty} \int_{2^{k+1}I/2^kI} \varphi_2 \omega_J dx|^2\| \\
&\leq \| \sum_{J\subset I} (\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \int_{2^{k+1}I/2^kI} |\varphi_2|^2)(\sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^kI} |\omega_J|^2)\| \\
&\leq c(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \| \int_{2^{k+1}I} |\varphi - \varphi_{2I}|^2\|) \\
&\qquad (\sum_{J\subset I} \sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^kI} |\omega_J|^2) \\
&\leq c|I| \|\varphi\|^2_{\mathcal{BMO}_c} \sum_{j=0}^{\infty} 2^j \sum_{k=1}^{\infty} \int_{2^{k+1}I/2^kI} 2^{2k} \frac{|2^{-j}I|^3}{|2^kI|^4} \\
&\leq c|I|
\end{aligned}$$

Therefore $\| \sum_{J\subset I} |\langle \varphi, \omega_J\rangle|^2\| \leq c|I|$, which complete our proof. □

Combined with Theorem 2.2.3 and Theorem 2.3.2, we have the following corollary

Corollary 2.4.2. *For $1 \leq p < \infty$, we have*

$$\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) = H_p^c(\mathbb{R}, \mathcal{M}).$$

Similar equality hold for $\mathcal{H}_p^r$ and H_p^r, and $\mathcal{H}_p$ and H_p.

If $\mathcal{M} = \mathbb{C}$, $\mathcal{H}_1(\mathbb{R}, \mathbb{C})$ is just the usual Hardy space $H_1(\mathbb{R})$ of $\mathbb{R}$. $H_1(\mathbb{R})$ also has the following characterization:

$$H_1(\mathbb{R}) = \{f \in L_1(\mathbb{R}) : H(f) \in L_1(\mathbb{R})\},$$

where H is the Hilbert transform of $\mathbb{R}$. For any $f \in H_1(\mathbb{R})$,

$$\|f\|_{H_1(\mathbb{R})} \approx \|f\|_{L_1(\mathbb{R})} + \|H(f)\|_{L_1(\mathbb{R})}.$$

Thus $H_1(\mathbb{R})$ can be viewed as a subspace of $L_1(\mathbb{R}) \oplus_1 L_1(\mathbb{R})$. The latter direct sum has its natural operator structure as an L_1 space. This induce an operator space structure on $H_1(\mathbb{R})$. Although $(w_I)_{I\in\mathcal{D}}$ is a unconditional basis of $H_1(\mathbb{R})$, Ricard [75] (see also [76]) proved that $H_1(\mathbb{R})$ does not have complete unconditional basis. However, in noncommutative analysis, one can introduce another natural operator space structure on $H_1(\mathbb{R})$ as follows: $S_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$, where S_1 is the trace class on ℓ_2. Then we have the following result. Note that Ricard [76] obtained a similar result using Hilbert space techniques.

Corollary 2.4.3. *The complete orthogonal systems* $(w_I)_{I\in\mathcal{D}}$ *of* $L_2(\mathbb{R})$ *is a completely unconditional basis for* $H_1(\mathbb{R})$ *if we define the operator space structure imposed on* $H_1(\mathbb{R})$ *by* $S_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$.

Proof. Fix a finite subset $\mathcal{I} \subset \mathcal{D}$. Let $T_\varepsilon f \doteq \sum_{I\in\mathcal{I}} \varepsilon_I \langle f, w_I\rangle w_I$, where $\varepsilon_I = \pm 1$. By the definition of $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$ (with $\mathcal{M} = B(\ell_2)$), the orthogonality of $(w_I)_{I\in\mathcal{D}}$ yields immediately that

$$\begin{aligned}\|T_\varepsilon f\|_{\mathcal{H}_1^c} &= \Big\|\Big(\sum_{I\in\mathcal{I}} \frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}\Big\|_{L_1(\mathcal{N})}\\ &\le \Big\|\Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}\Big\|_{L_1(\mathcal{N})} = \|f\|_{\mathcal{H}_1^c}\end{aligned}$$

Similarly, the above inequality hold for $\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})$. Now, let $f \in \mathcal{H}_1(\mathbb{R}, \mathcal{M})$, then for any $\epsilon > 0$, there exists a decomposition $f = g + h$ such that

$$\|g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{M})} \le \|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} + \epsilon.$$

Therefore

$$\begin{aligned}\|T_\varepsilon f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} &\le \|T_\varepsilon g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|T_\varepsilon h\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})}\\ &\le \|g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{M})} \le \|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} + \epsilon.\end{aligned}$$

Let $\epsilon \to 0$, we get the result. $\square$

Chapter 3

Harmonic analysis on quantum tori

Introduction

The subject of this paper follows the current line of investigation on noncommutative harmonic analysis. This topic has many interactions with other fields such as operator spaces, quantum probability, operator algebras, and of course, harmonic analysis. The aspect we are interested in is particularly related to the recent developments of noncommutative martingale/ergodic inequalities and Littlewood-Paley-Stein theory for quantum Markov semigroups. Motivated by operator spaces and by using tools from this theory, many classical martingale and ergodic inequalities have been successfully transferred to the noncommutative setting (see, for instance, [69, 35, 43, 45, 73, 74, 63, 9, 4, 61]). These inequalities of quantum probabilistic nature have, in return, applications to operator space theory (cf., e.g. [67, 33, 40, 41, 42, 91, 92]). Closely related to that, harmonic analysis on qua ntum semigroups has started to be developed in the last years. This first period of development of the noncommutative Littlewood-Paley-Stein theory deals with square function inequalities, H_1-BMO duality and Riesz transforms (cf. [37, 52, 53, 34, 38]). One can also include in this topic the very fresh promising direction of research on the Calderón-Zygmund singular integral operators in the noncommutative setting (cf. [62, 54, 39]). The concern of the present paper is directly linked to this last direction. Our objective is to develop harmonic analysis on quantum tori.

Quantum or noncommutative tori are fundamental examples in operator algebras and probably the most accessible interesting class of objects of study in noncommutative geometry (cf. [14, 15]). There exist extensive works on them (see, for instance, the survey paper by Rieffel [77] for those before the 1990's). We refer to [16, 22, 85] for some illustrations of more recent developments on this topic.

We now recall the definition of quantum tori. Let $d \geq 2$ and $\theta = (\theta_{kj})$ be a real skew-symmetric $d \times d$-matrix. The d-dimensional noncommutative torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by d unitary operators $U_1, \dots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \quad j, k = 1, \dots, d.$$

There exists a faithful tracial state τ on $\mathcal{A}_\theta$. Let $\mathbb{T}^d_\theta$ be the von Neumann algebra in the GNS representation of τ. $\mathbb{T}^d_\theta$ is called the quantum d-torus associated with θ. Note that if $\theta = 0$, then $\mathcal{A}_\theta = C(\mathbb{T}^d)$ and $\mathbb{T}^d_\theta = L_\infty(\mathbb{T}^d)$, where $\mathbb{T}^d$ is the usual d-torus. So a quantum d-torus is a deformation of the usual d-torus. It is thus natural to expect that $\mathbb{T}^d_\theta$ shares many properties with $\mathbb{T}^d$. This is indeed the case for differential geometry, as shown by the works of Connes and his collaborators. However, little is done regarding analysis. To our best knowledge, up to now, only the mean convergence theorem of quantum Fourier series by the square Fejér summation was proved at the C^*-algebra level (cf. [86, 87]), and on the other hand, the quantum torus analogue of Sobolev inequalities was obtained only in the Hilbert, i.e., L_2 space case (cf. [80]). The reason of this lack of development of analysis might be explained by numerous di fficulties one may encounter when dealing with noncommutative L_p-spaces, since these spaces come up unavoidably if one wishes to do analysis on quantum tori. For instance, the usual way of proving pointwise convergence theorems is to pass through the corresponding maximal inequalities. But the study of maximal inequalities is one of the most delicate and difficult parts in noncommutative analysis.

This paper is the first one of a long project that intends to develop analysis on quantum tori and more generally on twisted crossed products by amenable groups. Our aim here is to study some important aspects of harmonic analysis on $\mathbb{T}^d_\theta$. The subject that we address is three-fold:

i) *Convergence of Fourier series.* We consider several summation methods on $\mathbb{T}^d_\theta$, including the square Fejér means, square and circular Poisson means, and Bochner-Riesz means. We first establish the maximal inequalities for them and then obtain the corresponding pointwise convergence theorems. This part heavily relies on the theory of noncommutative martingale and ergodic inequalities.

ii) *Fourier multipliers.* The right framework for our study of Fourier multipliers is operator space theory. We show that for $1 \le p \le \infty$ the completely bounded L_p Fourier multipliers on $\mathbb{T}^d_\theta$ coincide with those on $\mathbb{T}^d$.

iii) *Hardy and* BMO *spaces.* Based on the recent development of the noncommutative Littlewood-Paley-Stein theory and the operator-valued harmonic analysis, we define Hardy and BMO spaces on $\mathbb{T}^d_\theta$ via the circular Poisson semigroup. We show that the properties of Hardy spaces in the classical case remain true in the quantum setting. In particular, we get the H_1-BMO duality theorem.

One main strategy for approaching these problems is to transfer them to the corresponding ones in the case of operator-valued functions on the classical tori, and then to use existing results in the latter case or adapt classical arguments. Due to the noncommutativity of operator product, substantial difficulties arise in our arguments, like usually in noncommutative analysis. One of the subtlest parts of our arguments is the proof of the weak type $(1, 1)$ maximal inequalities for the square Fejér and Poisson means because of their multiple-parameter nature. This is the first time that noncommutative weak type $(1, 1)$ maximal inequalities are considered for mappings of this nature. Another intricate part concerns the analogue for $\mathbb{T}^d_\theta$ of the classical Stein theorem on Bochner-Riesz means. The proof of the corresponding maximal inequalities is quite technical too. Our study of Hardy spaces via the Littlewood-Paley theory necessitates a very careful analysis of various BMO-norms and square functions. The difficulty of this study is partly explained by the lack of an explicit handy formula of the circular Poisson kernel on $\mathbb{T}^d$ for $d \ge 2$.

We end this introduction with a brief description of the organization of the paper. In Section 3.1 we present some preliminaries and notation on quantum tori. This section also introduces our transference method. The simple section 3.2 defines the summation methods studied in the paper and deals with the mean convergence of quantum Fourier series by them. Section 4 is devoted to the maximal inequalities associated to these summation methods. Their proofs depend, via transference, on some general maximal inequalities for operator-valued functions on $\mathbb{R}^d$ (or $\mathbb{T}^d$) that are of interest for their own right. These maximal inequalities are then applied in Section 3.4 to obtain the corresponding pointwise convergence theorems. Section 3.5 deals with the Bochner-Riesz means. The main theorem there is the quantum analogue of Stein's classical theorem. The difficult part is the type (p,p) maximal inequality for these means. In Section 3.6 we discuss L_p Fourier multipliers on $\mathbb{T}^d_\theta$. We show that a Fourier multiplier is completely bounded on the noncommutative L_p-space associated to $\mathbb{T}^d_\theta$ iff it is completely bounded on $L_p(\mathbb{T}^d)$. In this case, the two completely bounded norms are equal. Finally, in Section 3.7, we present the Littlewood-Paley theory on $\mathbb{T}^d_\theta$ and define the associated Hardy and BMO spaces using the circular Poisson semigroup, and show that they possess all expected properties of the usual Hardy spaces on $\mathbb{R}^d$. Our approach is to transfer this theory to the operator-valued case on $\mathbb{T}^d$ and to use Mei's arguments in [52] for the $\mathbb{R}^d$ setting. Since the geometry of $\mathbb{T}^d$ and the circular Poisson kernel are less handy than those of $\mathbb{R}^d$, we cannot directly apply Mei's results to our case. However, considering functions on $\mathbb{T}^d$ as periodic functions on $\mathbb{R}^d$, we can still reduce most of our problems to the corresponding ones on periodic functions on $\mathbb{R}^d$, then adapt Mei's argument to the periodic case. A good part of this section is devoted to the study of several BMO-norms and square functions naturally appearing in this periodization procedure.

3.1 Preliminaries

3.1.1 Noncommutative L_p spaces

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}_+$ its positive part. Recall that a *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0,\infty]$ satisfying:

i) $\tau(x+y) = \tau(x) + \tau(y)$ for arbitrary $x, y \in \mathcal{M}_+$;

ii) $\tau(\lambda x) = \lambda\tau(x)$ for any $\lambda \in [0,\infty)$ and $x \in \mathcal{M}_+$;

iii) $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathcal{M}$.

τ is said to be *normal* if $\sup_\gamma \tau(x_\gamma) = \tau(\sup_\gamma x_\gamma)$ for any bounded increasing net (x_γ) in $\mathcal{M}_+$, *semifinite* if for each $x \in \mathcal{M}_+\backslash\{0\}$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \le x$ and $\tau(y) < \infty$, and *faithful* if for each $x \in \mathcal{M}_+\backslash\{0\}$, $\tau(x) > 0$. A von Neumann algebra $\mathcal{M}$ is called *semifinite* if it admits a normal semifinite faithful trace τ. We refer to [84] for theory of von Neumann algebras. Throughout this paper, $\mathcal{M}$ will always denote a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ.

Denote by $\mathcal{S}_+$ the set of all $x \in \mathcal{M}_+$ such that $\tau(\mathrm{supp}(x)) < \infty$, where $\mathrm{supp}(x)$ is the support of x which is defined as the least projection e in $\mathcal{M}$ such that $ex = x$ or equivalently $xe = x$. Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Then $\mathcal{S}$ is a $*$-subalgebra of $\mathcal{M}$ which is w^*-dense in $\mathcal{M}$. Moreover, for each $0 < p < \infty$, $x \in \mathcal{S}$ implies $|x|^p \in \mathcal{S}_+$ (and so $\tau(|x|^p) < \infty$), where $|x| = (x^*x)^{1/2}$ is the modulus of x. Now, we define $\|x\|_p = [\tau(|x|^p)]^{1/p}$ for all $x \in \mathcal{S}$. One can show that $\|\cdot\|_p$ is a norm on $\mathcal{S}$ if $1 \le p < \infty$, and a quasi-norm (more precisely, p-norm) if $0 < p < 1$. The completion of $(\mathcal{S}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M},\tau)$ or simply by $L_p(\mathcal{M})$. This is the noncommutative L_p-space associated with $(\mathcal{M},\tau)$. The elements of $L_p(\mathcal{M})$ can be described by densely defined closed operators measurable with respect to $(\mathcal{M},\tau)$, like in the commutative case. For convenience, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. The trace τ can be extended to a linear functional on

$\mathcal{S}$, still denoted by τ. Since $|\tau(x)| \le \|x\|_1$ for all $x \in \mathcal{S}$, τ further extends to a continuous functional on $L_1(\mathcal{M})$.

Let $0 < r, p, q \le \infty$ be such that $1/r = 1/p + 1/q$. If $x \in L_p(\mathcal{M}), y \in L_q(\mathcal{M})$ then $xy \in L_r(\mathcal{M})$ and the following Hölder inequality holds:

$$\|xy\|_r \le \|x\|_p \|y\|_q.$$

In particular, if $r = 1$, $|\tau(xy)| \le \|xy\|_1 \le \|x\|_p \|y\|_q$ for arbitrary $x \in L_p(\mathcal{M})$ and $y \in L_q(\mathcal{M})$. This defines a natural duality between $L_p(\mathcal{M})$ and $L_q(\mathcal{M}) : \langle x, y\rangle = \tau(xy)$. For any $1 \le p < \infty$ we have $L_p(\mathcal{M})^* = L_q(\mathcal{M})$ isometrically. Thus, $L_1(\mathcal{M})$ is the predual $\mathcal{M}_*$ of $\mathcal{M}$, and $L_p(\mathcal{M})$ is reflexive for $1 < p < \infty$. We refer to [70] for more information on noncommutative L_p-spaces.

3.1.2 Quantum tori

Let $d \ge 2$ and $\theta = (\theta_{kj})$ be a real skew symmetric $d \times d$-matrix. The associated d-dimensional noncommutative torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by d unitary operators $U_1, \dots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \quad j, k = 1, \dots, d. \tag{3.1.1}$$

We will use standard notation from multiple Fourier series. Let $U = (U_1, \cdots, U_d)$. For $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$ we define

$$U^m = U_1^{m_1} \cdots U_d^{m_d}.$$

A polynomial in U is a finite sum

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \quad \text{with} \quad \alpha_m \in \mathbb{C},$$

that is, $\alpha_m = 0$ for all but finite indices $m \in \mathbb{Z}^d$. The involution algebra $\mathcal{P}_\theta$ of such all polynomials is dense in $\mathcal{A}_\theta$. For any polynomial x as above we define

$$\tau(x) = \alpha_{\mathbf{0}},$$

where $\mathbf{0} = (0, \cdots, 0)$. Then, τ extends to a faithful tracial state on $\mathcal{A}_\theta$. Let $\mathbb{T}_\theta^d$ be the w^*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ. This is our d-dimensional quantum torus. The state τ extends to a normal faithful tracial state on $\mathbb{T}_\theta^d$ that will be denoted again by τ. Recall that the von Neumann algebra $\mathbb{T}_\theta^d$ is hyperfinite.

Since τ is a state, $L_q(\mathbb{T}_\theta^d) \subset L_p(\mathbb{T}_\theta^d)$ for any $0 < p < q \le \infty$. Any $x \in L_1(\mathbb{T}_\theta^d)$ admits a formal Fourier series:

$$x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m,$$

where

$$\hat{x}(m) = \tau((U^m)^* x), \quad m \in \mathbb{Z}^d$$

are the Fourier coefficients of x. x is, of course, uniquely determined by its Fourier series.

3.1.3 Transference

We denote the usual d-torus by $\mathbb{T}^d$:

$$\mathbb{T}^d = \{(z_1, \ldots, z_d) : \ |z_j| = 1, z_j \in \mathbb{C}, \ 1 \le j \le d\}.$$

$\mathbb{T}^d$ is equipped with the usual topology and group law multiplication, that is,

$$z \cdot w = (z_1, \ldots, z_d) \cdot (w_1, \ldots, w_d) = (z_1 w_1, \ldots, z_d w_d).$$

For any $m \in \mathbb{Z}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ let

$$z^m = z_1^{m_1} \cdots z_d^{m_d}.$$

We will need the tensor von Neumann algebra $\mathcal{N}_\theta = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathbb{T}^d_\theta$, equipped with the tensor trace $\nu = \int dm \otimes \tau$, where dm is normalized Haar measure on $\mathbb{T}^d$. Note that for every $0 < p < \infty$,

$$L_p(\mathcal{N}_\theta, \nu) \cong L_p(\mathbb{T}^d; L_p(\mathbb{T}^d_\theta)).$$

The space on the right hand side is the space of Bochner p-integrable functions from $\mathbb{T}^d$ to $L_p(\mathbb{T}^d_\theta)$. Accordingly, let $C(\mathbb{T}^d; \mathcal{A}_\theta)$ denote the C^*-algebra of continuous functions from $\mathbb{T}^d$ to $\mathcal{A}_\theta$. For each $z \in \mathbb{T}^d$, define π_z to be the isomorphism of $\mathbb{T}^d_\theta$ determined by

$$\pi_z(U^m) = z^m U^m = z_1^{m_1} \cdots z_d^{m_d} U_1^{m_1} \cdots U_d^{m_d}.$$

It is clear that π_z is trace preserving, so extends to an isometry on $L_p(\mathbb{T}^d_\theta)$ for every $0 < p < \infty$. Thus we have

$$\|\pi_z(x)\|_p = \|x\|_p, \quad x \in L_p(\mathbb{T}^d_\theta), \ 0 < p \le \infty.$$

Proposition 3.1.1. *For any $x \in L_p(\mathbb{T}^d_\theta)$ the function $\tilde{x} : z \mapsto \pi_z(x)$ is continuous from $\mathbb{T}^d$ to $L_p(\mathbb{T}^d_\theta)$ (with respect to the w^*-topology for $p = \infty$). If $x \in \mathcal{A}_\theta$, it is continuous from $\mathbb{T}^d$ to $\mathcal{A}_\theta$.*

Proof. Consider first the case $0 < p < \infty$. Let $x \in L_p(\mathbb{T}^d_\theta)$. Since $\mathcal{P}_\theta$ is dense in $L_p(\mathbb{T}^d_\theta)$, for arbitrary $\varepsilon > 0$ there is $x_0 \in \mathcal{P}_\theta$ such that $\|x - x_0\|_p < \varepsilon$. Clearly, $\pi_z(x_0)$ is a polynomial in U of the same degree as x_0. Thus, $z \mapsto \pi_z(x_0)$ is continuous from $\mathbb{T}^d$ into $L_p(\mathbb{T}^d_\theta)$. We then deduce the desired continuity of $\tilde{x}$. The same argument works equally for $\mathcal{A}_\theta$. The case of $p = \infty$ follows from that of $p = 1$ by duality. □

The previous result in the case of $p = \infty$ implies, in particular, that the map $x \mapsto \tilde{x}$ establishes an isomorphism from $\mathbb{T}^d_\theta$ into $\mathcal{N}_\theta$. It is also clear that this isomorphism is trace preserving. Thus we get the following

Corollary 3.1.2. i) *Let $0 < p \le \infty$. If $x \in L_p(\mathbb{T}^d_\theta)$, then $\tilde{x} \in L_p(\mathcal{N}_\theta)$ and $\|\tilde{x}\|_p = \|x\|_p$, that is, $x \mapsto \tilde{x}$ is an isometric embedding from $L_p(\mathbb{T}^d_\theta)$ into $L_p(\mathcal{N}_\theta)$. Moreover, this map is also an isomorphism from $\mathcal{A}_\theta$ into $C(\mathbb{T}^d; \mathcal{A}_\theta)$.*

ii) *Let $\widetilde{\mathbb{T}^d_\theta} = \{\tilde{x} : x \in \mathbb{T}^d_\theta\}$. Then $\widetilde{\mathbb{T}^d_\theta}$ is a von Neumann subalgebra of $\mathcal{N}_\theta$ and the associated conditional expectation is given by*

$$\mathbb{E}(f)(z) = \pi_z\Big(\int_{\mathbb{T}^d} \pi_{\bar{w}}[f(w)]dm(w)\Big), \quad z \in \mathbb{T}^d, \ f \in \mathcal{N}_\theta.$$

Moreover, $\mathbb{E}$ extends to a contractive projection from $L_p(\mathcal{N}_\theta)$ onto $L_p(\widetilde{\mathbb{T}^d_\theta})$ for $1 \le p \le \infty$.

iii) *$L_p(\mathbb{T}^d_\theta)$ is isometric to $L_p(\widetilde{\mathbb{T}^d_\theta})$ for every $0 < p \leq \infty$.*

Our transference method consists in the following procedure:

$$x \in L_p(\mathbb{T}^d_\theta) \mapsto \tilde{x} \in L_p(\widetilde{\mathbb{T}^d_\theta}) \subset L_p(\mathcal{N}_\theta).$$

This allows us to work in $L_p(\mathcal{N}_\theta)$. Then, in order to return back to $L_p(\widetilde{\mathbb{T}^d_\theta}) \cong L_p(\mathbb{T}^d_\theta)$, we apply the conditional expectation $\mathbb{E}$ to elements in $L_p(\mathcal{N}_\theta)$.

3.2 Mean Convergence

We begin with the mean convergence of Fourier series defined on quantum tori for an illustration of the transference method described in the previous section. This section also introduces the summation methods studied throughout the paper. They are the following:

- The *square Fejér mean*

$$F_N[x] = \sum_{m \in \mathbb{Z}^d, |m|_\infty \leq N} \Big(1 - \frac{|m_1|}{N+1}\Big) \cdots \Big(1 - \frac{|m_d|}{N+1}\Big) \hat{x}(m) U^m, \quad N \geq 0.$$

- The *square Poisson mean*

$$P_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_1} U^m, \quad 0 \leq r < 1.$$

- The *circular Poisson mean*

$$\mathbb{P}_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m, \quad 0 \leq r < 1.$$

- Let Φ be a continuous function on $\mathbb{R}^d$ with $\Phi(0) = 1$. Define

$$\Phi^\varepsilon[x] = \sum_{m \in \mathbb{Z}^d} \Phi(\varepsilon m) \hat{x}(m) U^m, \quad \varepsilon > 0.$$

We will always impose the following condition to Φ:

$$\begin{cases} \Phi(s) = \hat{\varphi}(s) \quad \text{with} \displaystyle\int_{\mathbb{R}^d} \varphi(s) ds = 1; \\ |\Phi(s)| + |\varphi(s)| \leq A(1 + |s|)^{-d-\delta}, \ \forall s \in \mathbb{R}^d, \end{cases} \tag{3.2.1}$$

for some $A, \delta > 0$ (cf. [83, p. 253]).
In the above, $x \in L_1(\mathbb{T}^d_\theta)$ has its Fourier series expansion: $x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m$, and for $m \in \mathbb{Z}^d$

$$|m|_p = (\sum_{j=1}^d |m_j|^p)^{1/p}$$

with the usual modification for $p = \infty$.

The last summation method contains two special important examples of the function Φ. The first one is

$$\Phi(s) = e^{-2\pi|s|} \quad \text{and} \quad \varphi(s) = c_d(1+|s|^2)^{-(d+1)/2}, \quad \forall s \in \mathbb{R}^d,$$

where we have used the standard notation in harmonic analysis that $|s| = |s|_2$ denotes the Euclidean norm of $\mathbb{R}^d$. In this case,

$$\Phi^\varepsilon[x] = \sum_{m\in\mathbb{Z}^d} e^{-2\pi|m|_2\varepsilon}\hat{x}(m)U^m.$$

This is the circular Poisson integral $\mathbb{P}_r[x]$ of x with $r = e^{-2\pi\varepsilon}$.

The second example arises when $\alpha > (d-1)/2$ in the following definition

$$\Phi(s) = \begin{cases} (1-|s|^2)^\alpha & \text{if} \quad |s| < 1, \\ 0 & \text{if} \quad |s| \geq 1. \end{cases}$$

It is well known that

$$\varphi(s) = \hat{\Phi}(s) = \frac{\Gamma(\alpha+1)J_{\frac{d}{2}+\alpha}(2\pi|s|)}{\pi^\alpha|s|^{\frac{d}{2}+\alpha}}, \quad \forall\, s \in \mathbb{R}^d \setminus \{0\},$$

where J_λ is the Bessel function of order λ. In this case we obtain the *Bochner-Riesz mean* of order α on the quantum torus:

$$B_R^\alpha[x] = \sum_{|m|_2\leq R} \Big(1 - \frac{|m|_2^2}{R^2}\Big)^\alpha \hat{x}(m)U^m.$$

A fundamental problem is in which sense the above means of the operator x converge back to x. This problem is partly investigated in this section. Indeed, we have the following mean convergence theorem.

Proposition 3.2.1. *Let $1 \leq p < \infty$ and $x \in L_p(\mathbb{T}_\theta^d)$. Then $F_N[x]$ converges to x in $L_p(\mathbb{T}_\theta^d)$ as $N \to \infty$. The same convergence holds for $P_r[x]$, $\mathbb{P}_r[x]$ as $r \to 1$ and $\Phi^\varepsilon[x]$ as $\varepsilon \to 0$. Moreover, for $p = \infty$ these limits hold for any $x \in \mathcal{A}_\theta$.*

The proof can be done either by imitating the classical proofs (cf. [83]), or using the transference argument. The second method is more elegant and simpler. The corresponding results in $L_p(\mathcal{N}_\theta)$ are simple and well-known when one writes $L_p(\mathcal{N}_\theta) = L_p(\mathbb{T}^d; L_p(\mathbb{T}_\theta^d))$, which reduces the mean convergence in $L_p(\mathbb{T}_\theta^d)$ to the corresponding one in the vector-valued case on the usual torus $\mathbb{T}^d$.

As all these summation methods in the vector-valued case are given by approximation identities, it is better to state and prove first a general convergence theorem for convolution operators by an approximation identity in $L_p(\mathbb{T}^d; X)$, where X is a Banach space. Here $L_p(\mathbb{T}^d; X)$ denotes the L_p-space of Bochner p-integrable functions from $\mathbb{T}^d$ to X.

Let Λ be a directed set. An *approximation identity* on the multiplication group $\mathbb{T}^d$ (as $\lambda \to \lambda_0$) is a family of functions $(\varphi_\lambda)_{\lambda\in\Lambda}$ in $L_1(\mathbb{T}^d)$ verifying the following three conditions:

i) $\int_{\mathbb{T}^d} \varphi_\lambda(z)dm(z) = 1$ for all $\lambda \in \Lambda$.

ii) $\sup_{\lambda\in\Lambda} \|\varphi_\lambda\|_1 < \infty$.

iii) For any neighborhood V of the identity $(1,\dots,1)$ of the group $\mathbb{T}^d$ we have

$$\int_{\mathbb{T}^d\setminus V} |\varphi_\lambda| dm(z) \to 0 \text{ as } \lambda \to \lambda_0.$$

Recall that for $N \geq 0$ an integer, the square Fejér kernel on $\mathbb{T}^d$ is

$$F_N(z) = \sum_{m\in\mathbb{Z}^d,\, |m|_\infty \leq N} \Big(1 - \frac{|m_1|}{N+1}\Big)\cdots\Big(1 - \frac{|m_d|}{N+1}\Big) z^m. \tag{3.2.2}$$

For $0 \leq r < 1$, the *square and circular* Poisson kernels are respectively

$$P_r(z) = \sum_{m\in\mathbb{Z}^d} r^{|m|_1} z^m \quad \text{and} \quad \mathbb{P}_r(z) = \sum_{m\in\mathbb{Z}^d} r^{|m|_2} z^m. \tag{3.2.3}$$

It is well known that $(F_N)_{N\geq 1}$, $(P_r)_{0\leq r<1}$ and $(\mathbb{P}_r)_{0\leq r<1}$ are all approximation identities on $\mathbb{T}^d$. Also, if we write $\Phi^\varepsilon(s) = \Phi(\varepsilon s)$, then $\Phi^\varepsilon = \widehat{\varphi_\varepsilon}$ with $\varphi_\varepsilon(s) = \frac{1}{\varepsilon^d}\varphi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}^d$. Let

$$K_\varepsilon(s) = \sum_{m\in\mathbb{Z}^d} \varphi_\varepsilon(s+m), \quad s \in \mathbb{R}^d.$$

K_ε is periodic, so can be viewed as a function on $\mathbb{T}^d$. Then by (3.2.1) it can be proved that $(K_\varepsilon)_{\varepsilon>0}$ is an approximation identity on $\mathbb{T}^d$ such that

$$(K_\varepsilon * f)(z) = \sum_{m\in\mathbb{Z}^d} \Phi(\varepsilon m)\hat{f}(m) z^m, \quad f \sim \sum_{m\in\mathbb{Z}^d} \hat{f}(m) z^m \tag{3.2.4}$$

(see the proof of Theorem VII.2.11 in [83]).

Let X be a Banach space and let $1 \leq p \leq \infty$. Suppose that $(\varphi_\lambda)_{\lambda\in\Lambda}$ is an approximation identity on $\mathbb{T}^d$. For any $f \in L_p(\mathbb{T}^d; X)$ we define the convolution $\varphi_\lambda * f$ by

$$(\varphi_\lambda * f)(z) = \int_{\mathbb{T}^d} f(w)\varphi_\lambda(\bar{w}\cdot z) dm(w), \quad \forall\, z \in \mathbb{T}^d.$$

Then for any $f \in L_p(\mathbb{T}^d; X)$ we have $\varphi_\lambda * f \in L_p(\mathbb{T}^d; X)$ and

$$\|\varphi_\lambda * f\|_p \leq \|f\|_p \|\varphi_\lambda\|_1.$$

The following vector-valued result is well-known. The proof in the scalar case (cf. e.g. [26, Theorem 1.2.19]) is valid as well in the vector-valued setting without any change. $C(\mathbb{T}^d; X)$ denotes the space of continuous functions from $\mathbb{T}^d$ to X, equipped with the uniform norm.

Proposition 3.2.2. *Let X be a Banach space and let $1 \leq p < \infty$. Let $(\varphi_\lambda)_{\lambda\in\Lambda}$ be an approximation identity on $\mathbb{T}^d$. If $f \in L_p(\mathbb{T}^d; X)$, then*

$$\|\varphi_\lambda * f - f\|_p \to 0 \text{ as } \lambda \to \lambda_0.$$

Moreover, when $p = \infty$ the above limit holds for any $f \in C(\mathbb{T}^d; X)$.

It is now clear that Proposition 3.2.1 immediately follows from Proposition 3.2.2 via the transference method.

3.3 Maximal inequalities

In this section, we present the maximal inequalities of the summation methods of Fourier series defined previously. These inequalities will be used for the pointwise convergence in the next section. We first recall the definition of the noncommutative maximal norm introduced by Pisier [65] and Junge [35]. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $1 \le p \le \infty$. We define $L_p(\mathcal{M}; \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n\ge 1}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n)$ in $L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad \forall\, n \ge 1.$$

The norm of x in $L_p(\mathcal{M}; \ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\big\{\|a\|_{2p} \sup_{n\ge 1} \|y_n\|_\infty \,\|b\|_{2p}\big\},$$

where the infimum runs over all factorizations of x as above.

We will follow the convention adopted in [45] that $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$ is denoted by $\big\|\sup_n^+ x_n\big\|_p$. We should warn the reader that $\big\|\sup_n^+ x_n\big\|_p$ is just a notation since $\sup_n x_n$ does not make any sense in the noncommutative setting. We find, however, that $\big\|\sup_n^+ x_n\big\|_p$ is more intuitive than $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$. The introduction of this notation is partly justified by the following remark.

Remark 3.3.1. Let $x = (x_n)$ be a sequence of selfadjoint operators in $L_p(\mathcal{M})$. Then $x \in L_p(\mathcal{M}; \ell_\infty)$ iff there exists a positive element $a \in L_p(\mathcal{M})$ such that $-a \le x_n \le a$ for all $n \ge 1$. In this case we have

$$\big\|{\sup_{n\ge 1}}^+ x_n\big\|_p = \inf\big\{\|a\|_p \;:\; a \in L_p(\mathcal{M}),\ -a \le x_n \le a,\ \forall\, n \ge 1\big\}.$$

More generally, if Λ is any index set, we define $L_p(\mathcal{M}; \ell_\infty(\Lambda))$ as the space of all $x = (x_\lambda)_{\lambda\in\Lambda}$ in $L_p(\mathcal{M})$ that can be factorized as

$$x_\lambda = a y_\lambda b \quad \text{with} \quad a, b \in L_{2p}(\mathcal{M}),\ y_\lambda \in L_\infty(\mathcal{M}),\ \sup_\lambda \|y_\lambda\|_\infty < \infty.$$

The norm of $L_p(\mathcal{M}; \ell_\infty(\Lambda))$ is defined by

$$\big\|{\sup_{\lambda\in\Lambda}}^+ x_\lambda\big\|_p = \inf_{x_\lambda = a y_\lambda b} \big\{\|a\|_{2p} \sup_{\lambda\in\Lambda} \|y_\lambda\|_\infty \,\|b\|_{2p}\big\}.$$

It is shown in [45] that $x \in L_p(\mathcal{M}; \ell_\infty(\Lambda))$ iff

$$\sup\big\{\big\|{\sup_{\lambda\in J}}^+ x_\lambda\big\|_p \;:\; J \subset \Lambda,\ J \text{ finite}\big\} < \infty.$$

In this case, $\big\|{\sup_{\lambda\in\Lambda}}^+ x_\lambda\big\|_p$ is equal to the above supremum.

The following is the main theorem of this section.

Theorem 3.3.2. i) *Let $x \in L_1(\mathbb{T}^d_\theta)$. Then for any $\alpha > 0$ there exists a projection $e \in \mathbb{T}^d_\theta$ such that*

$$\sup_{N\ge 0} \big\|eF_N[x]e\big\|_\infty \le \alpha \quad \textit{and} \quad \tau(e^\perp) \le C_d \,\frac{\|x\|_1}{\alpha}.$$

ii) *Let* $1 < p \leq \infty$. *Then*

$$\big\|\sup_{N\geq 0}{}^{+} F_N[x]\big\|_p \leq C_d \,\frac{p^2}{(p-1)^2}\, \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}^d_\theta).$$

Both statements hold for the three other summation methods P_r, $\mathbb{P}_r$ *and* Φ^ε. *In the case of* Φ^ε, *the constant* C_d *also depends on the two constants in* (3.2.1).

In the terminology of [45], we can rephrase parts i) and ii) as that the map $x \mapsto (F_N[x])_{N\geq 0}$ is of weak type $(1,1)$ and of type (p,p), respectively. Before proceeding to the proof of the theorem, we point out that its part concerning the circular Poisson mean $\mathbb{P}_r$ can be easily deduced from [45]. This is due to the fact that $(\mathbb{P}_r)_{0\leq r<1}$ is a symmetric diffusion semigroup on $\mathbb{T}^d_\theta$. Let us show this latter statement. Define

$$\delta_j(U_j) = 2\pi \mathrm{i} U_j, \quad \delta_j(U_k) = 0, \quad k \neq j$$

(cf. [14]). These operators δ_j commute with the involution of $\mathbb{T}^d_\theta$ and play the role of the partial derivatives $\frac{\partial}{\partial x_j}$ on the classical d-torus. Let $\triangle = \sum_{j=1}^d \delta_j^2$. Then $\triangle$ is a negative operator on $L_2(\mathbb{T}^d_\theta)$ and its spectrum consists of the numbers $-4\pi^2|m|_2^2$, $m \in \mathbb{Z}^d$. For any $\lambda > 0$, we have

$$\|(\lambda - \triangle)^{-1}\| \leq \sup_{z\in\sigma(-\triangle)} \frac{1}{|\lambda + z|} \leq \frac{1}{\lambda}.$$

Then by the Hille-Yosida theorem, $\triangle$ is the infinitesimal generator of a semigroup of contractions on $L_2(\mathbb{T}^d_\theta)$. Denote this semigroup by (T_t). Then $T_t = \exp(t\triangle)$. It is easy to check that (T_t) satisfies the following properties:

i) T_t is a contraction on $\mathbb{T}^d_\theta : \|T_t x\|_\infty \leq \|x\|_\infty$ for all $x \in \mathbb{T}^d_\theta$;

ii) T_t is positive: $T_t x \geq 0$ if $x \geq 0$;

iii) $\tau \circ T_t = \tau : \tau(T_t x) = \tau(x)$ for all $x \in \mathbb{T}^d_\theta$;

iv) T_t is symmetric relative to $\tau : \tau(T_t(y)^* x) = \tau(y^* T_t(x))$ for all $x, y \in L_2(\mathbb{T}^d_\theta)$.

Then (T_t) extends to a semigroup of contractions on $L_p(\mathbb{T}^d_\theta)$ for every $1 \leq p \leq \infty$. This is the heat semigroup of $\mathbb{T}^d_\theta$. The circular Poisson means $\mathbb{P}_r[x]$ is exactly the Poisson semigroup subordinated to T_t, where $r = e^{-2\pi t}$. Then by [45], we get the part of Theorem 3.3.2 concerning the circular Poisson means.

The previous argument does not apply to the three other means. However, we can get the type (p,p) inequality for F_N and P_r again from [45] but not with the right estimate on the constant C_p. Indeed, the square Poisson mean P_r is the restriction to the diagonal $(r, ..., r)$ of the following multiple parameter semigroup $P_{(r_1,\dots,r_d)}$:

$$P_{(r_1,\dots,r_d)}[x] = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) r_1^{|m_1|} \cdots r_d^{|m_d|} U^m.$$

By iteration $P_{(r_1,\dots,r_d)}$ satisfies a maximal inequality on $L_p(\mathbb{T}^d_\theta)$ with a relevant constant controlled by $C^d p^{2d}/(p-1)^{2d}$. It then follows that the map $x \mapsto (P_r[x])_r$ is of type (p,p) with the same constant. Since each Fejér mean F_N is majorized by P_r for an appropriate r, we deduce that the same maximal inequality holds for F_N. We cannot, unfortunately, prove the weak type $(1,1)$ maximal inequality for F_N and P_r in this way.

The rest of this section is essentially devoted to the proof of Theorem 3.3.2. We will use transference and require the following two theorems which are of interest for their own right. Recall that $\mathcal{M}$ denotes a von Neumann algebra with a normal semifinite faithful trace τ. $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ is equipped with the tensor trace $\nu = dx \otimes \tau$, where dx is Lebesgue measure on $\mathbb{R}^d$.

Theorem 3.3.3. *Let φ be an integrable function on $\mathbb{R}^d$ such that $|\varphi|$ is radial and radially decreasing. Let $\varphi_\varepsilon(s) = \frac{1}{\varepsilon^d}\,\varphi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}^d$ and $\varepsilon > 0$.*

i) *Let $f \in L_1(\mathbb{R}^d; L_1(\mathcal{M}))$. Then for any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ such that*
$$\sup_{\varepsilon>0} \big\|e(\varphi_\varepsilon * f)e\big\|_\infty \le \alpha \quad \textit{and} \quad \nu(e^\perp) \le C_d\|\varphi\|_1 \frac{\|f\|_1}{\alpha}.$$

ii) *Let $1 < p \le \infty$. Then*
$$\big\|{\sup_{\varepsilon>0}}^{+}\varphi_\varepsilon * f\big\|_p \le C_d\|\varphi\|_1 \,\frac{p^2}{(p-1)^2}\|f\|_p, \quad \forall\, f \in L_p(\mathbb{R}^d; L_p(\mathcal{M})).$$

Proof. Let $f \in L_1(\mathbb{R}^d; L_1(\mathcal{M}))$. Without loss of generality, we assume that f is positive. On the other hand, it is easy to reduce the problem to the case where φ is positive too. Indeed, decomposing φ into its real and imaginary parts, we need only to consider each part separately. Since $f \ge 0$, we have
$$\mathrm{Re}(\varphi_\varepsilon) * f \le |\mathrm{Re}(\varphi_\varepsilon)| * f \le |\varphi|_\varepsilon * f.$$
This gives the announced reduction. Thus in the sequel we assume that $\varphi \ge 0$. First take φ to be of the form $\varphi = \sum_k \alpha_k \mathbb{1}_{B_k}$ (a finite sum), where B_k are balls of center 0 and $\alpha_k \ge 0$. Then
$$\varphi_\varepsilon * f(s) = \sum_k \alpha_k(\mathbb{1}_{B_k})_\varepsilon * f(s) = \sum_k \alpha_k|B_k| M_{\varepsilon B_k}(f)(s),$$
where $M_B(f)(s) = \frac{1}{|B|}\int_B f(s-t)dt$ for any ball B centered at 0. We now appeal to Mei's noncommutative Hardy-Littlewood maximal weak type (1,1) inequality [52]: For any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ such that
$$\nu(e^\perp) \le C_d\,\frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|eM_B(f)e\|_\infty \le \alpha,\ \forall \text{ ball } B \text{ centered at } 0.$$
We then deduce that
$$\|e(\varphi_\varepsilon * f)e\|_\infty \le C_d \sum_k \alpha_k|B_k|\alpha = C_d\|\varphi\|_1\alpha, \quad \forall\, \varepsilon > 0.$$
For a general positive φ, choose an increasing sequence $(\varphi^{(n)})$ of functions of the previous form such that $\varphi^{(n)}$ converges to φ pointwise. Then for any $\alpha > 0$, there exists a projection $e_n \in L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ such that
$$\nu(e_n^\perp) \lesssim \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|e_n(\varphi_\varepsilon^{(n)} * f)e_n\|_\infty \le \alpha, \quad \forall\, \varepsilon > 0.$$
Let a be a w^*-accumulation point of e_n. Note that
$$(\varphi_\varepsilon^{(n)} * f)^{\frac12}e_n - (\varphi_\varepsilon * f)^{\frac12}a = ((\varphi_\varepsilon^{(n)} * f)^{\frac12} - (\varphi_\varepsilon * f)^{\frac12})e_n + (\varphi_\varepsilon * f)^{\frac12}(e_n - a).$$

Since $\varphi^{(n)} \to \varphi$ increasingly, then $(\varphi_\varepsilon^{(n)} * f)^{\frac{1}{2}}$ strongly converges to $(\varphi_\varepsilon * f)^{\frac{1}{2}}$. Hence $(\varphi_\varepsilon^{(n)} * f)^{\frac{1}{2}} e_n$ weakly converges to $(\varphi_\varepsilon * f)^{\frac{1}{2}} a$. Then we deduce

$$\nu(1-a) \lesssim \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|(\varphi_\varepsilon * f)^{\frac{1}{2}} a\|_\infty \leq \liminf_n \|(\varphi_\varepsilon^{(n)} * f)^{\frac{1}{2}} e_n\|_\infty \leq \alpha^{\frac{1}{2}}.$$

Let $e = \mathbb{1}_{[\frac{1}{2},\, 1]}(a)$, the spectral projection of a corresponding to the interval $[\frac{1}{2},\, 1]$. Note that $1-e = \mathbb{1}_{[\frac{1}{2},\, 1]}(1-a)$. Then $\frac{1}{2}(1-e) \leq 1-a$, which implies that $\frac{1}{2}\nu(1-e) \leq \nu(1-a)$. Moreover, letting $g(r) = \frac{1}{r}\mathbb{1}_{[\frac{1}{2},\, 1]}(r),\ r \in (0,1]$, we have $e = eg(a)a$ and

$$e(\varphi_\varepsilon * f)e = eg(a)[a(\varphi_\varepsilon * f)a]eg(a).$$

Since $\|eg(a)\|_\infty \leq 2$, we deduce that

$$\|e(\varphi_\varepsilon * f)e\|_\infty \leq 4\|a(\varphi_\varepsilon * f)a\|_\infty \leq 4\alpha.$$

Therefore the projection e satisfies:

$$\nu(e^\perp) \lesssim \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|e(\varphi_\varepsilon * f)e\|_\infty \leq 4\alpha, \quad \forall\, \varepsilon > 0.$$

Thus we get i).

Part ii) is proved by interpolation. It is clear that the map $f \mapsto (\varphi_\varepsilon * f)_{\varepsilon>0}$ is of type (∞,∞) with constant $\|\varphi\|_1$. On the other hand, since we have assumed that $\varphi \geq 0$, $\varphi_\varepsilon * f \geq 0$ for $f \geq 0$. Thus by the interpolation theorem from [45], we deduce the desired (p,p) type maximal inequality, i.e., part ii). □

The conclusion of the previous theorem also holds for another family of functions φ which satisfy an estimate of multiple-parameter nature.

Theorem 3.3.4. *Let φ be an integrable function on $\mathbb{R}^d$ that has the following decomposition: $\varphi(s_1, \cdots, s_d) = \varphi_1(s_1)\cdots\varphi_d(s_d)$, where each φ_k satisfies*

$$|\varphi_k(t)| \leq \frac{A}{(1+|t|)^{1+\delta}}, \quad \forall t \in \mathbb{R},$$

for some $A, \delta > 0$. Then the conclusion of Theorem 3.3.3 remains true.

Proof. This proof is much more involved than the previous one. Again, we can assume that all functions φ_k are positive. It suffices to show the weak type $(1,1)$ inequality. Fix a positive $f \in L_1(\mathbb{R}^d; L_1(\mathcal{M}))$. Let $I_0 = [-1,\ 1]$ and $I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \leq 2^k\}$ for $k = 1, 2, \ldots$. Also, let $\tilde{I}_k = [-2^k,\ 2^k]$. Split $\mathbb{R}^d$ into $d!$ regions of the form $|t_{j_1}| \geq \cdots \geq |t_{j_d}|$, where $\{j_1, \ldots, j_d\}$ is a permutation of the set $\{1, \ldots, d\}$. Then

$$\varphi_\varepsilon * f(s) = \sum_{\{j_1,\ldots,j_d\}} \int_{|t_{j_1}|\geq\cdots\geq|t_{j_d}|} \varphi(t) f(s-\varepsilon t)dt.$$

By symmetry, it suffices to consider one of these regions, say the one where $|y_1| \geq \cdots \geq |y_d|$. Let

$$F_\varepsilon(s) = \int_{|t_1|\geq\cdots\geq|t_d|} \varphi(t) f(s-\varepsilon t)dt, \quad s = (s_1, ..., s_d) \in \mathbb{R}^d.$$

We must show that for any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ such that

$$\nu(e^\perp) \lesssim \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|eF_\varepsilon e\|_\infty \leq \alpha. \tag{3.3.1}$$

Using the assumption on φ and by change of variables, we have

$$\begin{aligned}
F_\varepsilon(s) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}}\int_{I_{k_1}}\int_{I_{k_2}}\cdots\int_{I_{k_d}}\varphi(t)f(s-\varepsilon t)dt\\
&\lesssim \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-k_1(1+\delta)}\dots 2^{-k_d(1+\delta)}\int_{I_{k_1}}\int_{I_{k_2}}\cdots\int_{I_{k_d}} f(s-\varepsilon t)dt\\
&\lesssim \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-k_1(1+\delta)}\dots 2^{-k_d(1+\delta)}\int_{\tilde I_{k_1}}\int_{\tilde I_{k_2}}\cdots\int_{\tilde I_{k_d}} f(s-\varepsilon t)dt\\
&\lesssim \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)\delta}\frac{1}{|\tilde I_{k_1}|^d}\int_{\tilde I_{k_1}^d} f(s_1-\varepsilon t_1, s_2-2^{k_2-k_1}\varepsilon t_2,\cdots, s_d-2^{k_d-k_1}\varepsilon t_d)dt.
\end{aligned}$$

Given a function $g\in L_1(\mathbb{R}^d;L_1(\mathcal{M}))$ and a cube $Q\subset\mathbb{R}^d$ centered at 0 and with sides parallel to the axes put

$$M_Q(g)(s)=\frac{1}{|Q|}\int_Q g(s-t)dt,\quad s\in\mathbb{R}^d.$$

Note that this average function appeared already in the proof of Theorem 3.3.3 but with balls instead of cubes. For any fixed $k=(k_1,\cdots,k_d)$ with $k_1\ge k_2\ge\cdots\ge k_d$ let

$$f_k(z_1,z_2,\cdots,z_d)=f(z_1,2^{k_2-k_1}z_2,\cdots,2^{k_d-k_1}z_d).$$

Then

$$\frac{1}{|\tilde I_{k_1}|^d}\int_{\tilde I_{k_1}^d} f(s_1-\varepsilon t_1, s_2-2^{k_2-k_1}\varepsilon t_2,\cdots, s_d-2^{k_d-k_1}\varepsilon t_d)dt = M_{\varepsilon\tilde I_{k_1}^d}(f_k)(s_1,2^{k_1-k_2}s_2,\cdots,2^{k_1-k_d}s_d).$$

Thus

$$F_\varepsilon(s)\lesssim \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)\delta}M_{\varepsilon\tilde I_{k_1}^d}(f_k)(s_1,2^{k_1-k_2}s_2,\cdots,2^{k_1-k_d}s_d).\tag{3.3.2}$$

Now we use again Mei's noncommutative Hardy-Littlewood maximal weak type $(1,1)$ inequality which remains true with balls replaced by cubes. For any $\alpha_k>0$, there exits a projection e_k in $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ such that

$$\nu(e_k^\perp)\le C_d\frac{\|f_k\|_1}{\alpha_k}\quad\text{and}\quad \|e_kM_{\varepsilon\tilde I_{k_1}^d}(f_k)e_k\|_\infty\le\alpha_k,\quad\forall\,\varepsilon>0.\tag{3.3.3}$$

Let T be the mapping

$$(s_1,s_2,\cdots,s_d)\mapsto(s_1,2^{k_1-k_2}s_2,\cdots,2^{k_1-k_d}s_d).$$

T is a homeomorphism of $\mathbb{R}^d$, so induces an isomorphism of $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, still denoted by T. Then for any $g\in L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, we have

$$\int\tau(T(g)(s))ds=\int\tau(g\circ T(s))ds=2^{k_2-k_1}\cdots 2^{k_d-k_1}\int\tau(g(s))ds.$$

Let $\tilde e_k=T(e_k)$. Then $\tilde e_k$ is a projection and

$$\nu(\tilde e_k^\perp)=2^{k_2-k_1}\cdots 2^{k_d-k_1}\nu(e_k^\perp).\tag{3.3.4}$$

On the other hand,

$$M_{\varepsilon \tilde{I}^d_{k_1}}(f_k)(s_1, 2^{k_1-k_2}s_2, \cdots, 2^{k_1-k_d}s_d) = T(M_{\varepsilon \tilde{I}^d_{k_1}}(f_k))(s_1, s_2, \cdots, s_d)$$

and

$$T(e_k M_{\varepsilon \tilde{I}^d_{k_1}}(f_k) e_k) = \tilde{e}_k M_{\varepsilon \tilde{I}^d_{k_1}}(f_k)(\cdot, 2^{k_1-k_2}\cdot, \cdots, 2^{k_1-k_d}\cdot)\tilde{e}_k.$$

Therefore, by (3.3.3)

$$\big\|\tilde{e}_k M_{\varepsilon \tilde{I}^d_{k_1}}(f_k)(\cdot, 2^{k_1-k_2}\cdot, \cdots, 2^{k_1-k_d}\cdot)\tilde{e}_k\big\|_\infty = \big\|e_k M_{\varepsilon \tilde{I}^d_{k_1}}(f_k) e_k\big\|_\infty \leq \alpha_k, \quad \forall\, \varepsilon > 0. \quad (3.3.5)$$

Let $\alpha > 0$. For each k with $k_1 \geq k_2 \geq \cdots \geq k_d$ we choose

$$\alpha_k = \alpha\, 2^{k_1\delta/(2d)} 2^{k_2\delta(1-1/(2d))} \cdots 2^{k_d\delta(1-1/(2d))}.$$

Then

$$2^{-(k_1+\cdots+k_d)\delta}\alpha_k = \alpha\, 2^{-k_1\delta/2}\, 2^{-n_2/(2d)} \cdots 2^{-n_d/(2d)}, \quad (3.3.6)$$

where $n_2 = k_1 - k_2, \cdots, n_d = k_1 - k_d$. Note that all n_j are nonnegative integers. Finally, let $e = \bigwedge_k \tilde{e}_k$. Then e is a projection in $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, and by (3.3.4), (3.3.3), the definition of f_k and the choice of α_k, we have

$$\begin{aligned}
\nu(e^\perp) &\leq \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} \nu(\tilde{e}_k^\perp) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{k_2-k_1}\cdots 2^{k_d-k_1}\nu(e_k^\perp) \\
&\leq C_d \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{k_2-k_1}\cdots 2^{k_d-k_1}\frac{\|f_k\|_1}{\alpha_k} \\
&\leq C_d\|f\|_1 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_d=0}^{\infty}\frac{1}{\alpha_k} \lesssim \frac{\|f\|_1}{\alpha}.
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, by (3.3.2), (3.3.5) and (3.3.6)

$$\begin{aligned}
\|eF_\varepsilon e\|_\infty &\leq \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)\delta}\big\|\tilde{e}_k M_{\varepsilon\tilde{I}^d_{k_1}}(f_k)(\cdot, 2^{k_1-k_2}\cdot, \cdots, 2^{k_1-k_d}\cdot)\tilde{e}_k\big\|_\infty \\
&\leq \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)\delta}\alpha_k \\
&\leq \alpha \sum_{k_1\geq 0}\sum_{n_2\geq 0}\cdots\sum_{n_d\geq 0} 2^{-k_1\delta/2}\, 2^{-n_2/(2d)}\cdots 2^{-n_d/(2d)} \lesssim \alpha.
\end{aligned}$$

Thus we get the desired estimate (3.3.1), so finish the proof of the theorem. □

We also require the following lemma for the proof of Theorem 3.3.2.

Lemma 3.3.5. *Let $\mathcal{N}$ be a w^*-closed involutive subalgebra of $\mathcal{M}$ that is the image of a normal conditional expectation $\mathcal{E}$. Let (x_n) be a sequence of positive operators in $L_1(\mathcal{N})$. Assume that for any $\alpha > 0$ there exists a projection $\tilde{e} \in \mathcal{M}$ such that*

$$\sup_n \|\tilde{e}x_n\tilde{e}\|_\infty \leq \alpha \quad \text{and} \quad \tau(\tilde{e}^\perp) \leq \frac{C}{\alpha}.$$

Then there exists a projection $e \in \mathcal{N}$ such that

$$\sup_n \|ex_ne\|_\infty \le 4\alpha \quad \textit{and} \quad \tau(e^\perp) \le \frac{2C}{\alpha}.$$

Proof. Let $a = \mathcal{E}(\tilde{e})$. Then $a \in \mathcal{N}$ and

$$\|ax_n^{1/2}\|_\infty = \|\mathcal{E}(\tilde{e}x_n^{1/2})\|_\infty \le \alpha^{1/2}.$$

As in the proof of Theorem 3.3.3, we then see that $e = \mathbb{1}_{[1/2,\,1]}(a)$ is the desired projection in $\mathcal{N}$. □

Proof of Theorem 3.3.2. We will identify the d-torus $\mathbb{T}^d$ with the cube $\mathbb{I}^d = [0,\,1]^d \subset \mathbb{R}^d$ (with $\mathbb{I} = [0,\,1]$) via $(e^{2\pi i s_1}, \cdots, e^{2\pi i s_d}) \leftrightarrow (s_1, \cdots, s_d)$. Accordingly, $\mathcal{N}_\theta = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathbb{T}^d_\theta$ is viewed as a subalgebra of $\mathcal{M}_\theta = L_\infty(\mathbb{R}^d)\overline{\otimes}\mathbb{T}^d_\theta$; the associated conditional expectation is just the multiplication by the indication function $\mathbb{1}_{\mathbb{I}^d}$ of $\mathbb{I}^d$. Thus $\widetilde{\mathbb{T}^d_\theta}$ becomes a subalgebra of $\mathcal{M}_\theta$ too. The corresponding conditional expectation is $\mathbb{1}_{\mathbb{I}^d} \cdot \mathbb{E}$, where $\mathbb{E}$ is the conditional expectation from $\mathcal{N}_\theta$ to $\widetilde{\mathbb{T}^d_\theta}$ given by Corollary 3.1.2.

Now let us show the weak type $(1,1)$ inequality for the Fejér means. Recall that F_N is the Fejér kernel on $\mathbb{T}^d$ given by (3.2.2) and that

$$F_N(s_1, \cdots, s_d) = G_N(s_1) \cdots G_N(s_d),$$

where G_N is the 1-dimensional Fejér kernel. It is a well-known elementary fact that

$$G_N(s) \le \frac{\pi^2}{2}\,\frac{N+1}{1+(N+1)^2|s|^2}.$$

Thus

$$F_N(s_1, \cdots, s_d) \lesssim \frac{1}{\varepsilon^d}\,\eta(\frac{s_1}{\varepsilon}) \cdots \eta(\frac{s_d}{\varepsilon}) = \eta_\varepsilon(s_1) \cdots \eta_\varepsilon(s_d),$$

where $\eta(s) = (1+|s|^2)^{-1}$ and $\varepsilon = (N+1)^{-1}$. Let $x \in L_1(\mathbb{T}^d_\theta)$. Writing x as a linear combination of four positive elements, we can assume $x \ge 0$. Using transference, we have that $\tilde{x} \in L_1(\widetilde{\mathbb{T}^d_\theta}) \subset L_1(\mathcal{N}_\theta)$ and

$$\begin{aligned}
\widetilde{F_N[x]}(s_1, \cdots, s_d) &= F_N * \tilde{x}(s_1, \cdots, s_d) \\
&= \int_{\mathbb{I}^d} F_N(s_1 - t_1, \cdots, s_d - t_d)\,\tilde{x}(t_1, \cdots, t_d)dt \\
&= \int_{\mathbb{R}^d} F_N(s_1 - t_1, \cdots, s_d - t_d)\mathbb{1}_{\mathbb{I}^d}(t_1, \cdots, t_d)\,\tilde{x}(t_1, \cdots, t_d)dt.
\end{aligned}$$

Therefore, we are in a situation of applying Theorem 3.3.4, so for any $\alpha > 0$ there exists a projection $\tilde{e} \in \mathcal{M}_\theta$ such that

$$\sup_N \|\tilde{e}\widetilde{F_N[x]}\tilde{e})\|_\infty \le \alpha \quad \text{and} \quad \nu(\tilde{e}^\perp) \lesssim \frac{\|\mathbb{1}_{\mathbb{I}^d}\tilde{x}\|_{L_1(\mathcal{M}_\theta)}}{\alpha} = \frac{\|x\|_1}{\alpha}.$$

Since $x \ge 0$, $\widetilde{F_N[x]} \ge 0$ for every N. Thus by Lemma 3.3.5, we get the desired weak type $(1,1)$ inequality for F_N. Similarly, we show the type (p,p) inequality. The same argument works equally for the square Poisson means P_r.

It remains to show the part of the theorem concerning Φ^ε (which contains the circular Poisson mean $\mathbb{P}_r$ as a special case). We will use the convolution formula (3.2.4). Note that for maximal inequalities on Φ^ε we do not need all conditions on Φ and φ in (3.2.1). What we really need here is the last growth assumption on φ there:

$$|\varphi(s)| \leq \frac{A}{(1+|s|)^{d+\delta}}, \quad s \in \mathbb{R}^d.$$

Then like in the proof of Theorem 3.3.3 we can assume that φ is nonnegative. In this case the kernel K_ε is nonnegative too. Moreover, replacing φ by the function on the right hand side above, we can further suppose that φ satisfies the assumption of Theorem 3.3.3. Now let $x \in L_1(\mathbb{T}_\theta^d)$. Without loss of generality, assume again $x \geq 0$. By (3.2.4), for $s = (s_1, \cdots, s_d) \in \mathbb{I}^d$ we have

$$\begin{aligned}
\widetilde{\Phi^\varepsilon[x]}(s) &= \int_{\mathbb{I}^d} K_\varepsilon(s-t)\,\tilde{x}(t)dt \\
&= \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{I}^d} \varphi_\varepsilon(s-t+m)\,\tilde{x}(t)dt \\
&= \int_{\mathbb{I}^d} \varphi_\varepsilon(s-t)\,\tilde{x}(t)dt + \sum_{m \neq \mathbf{0}} \int_{\mathbb{I}^d} \varphi_\varepsilon(s-t+m)\,\tilde{x}(t)dt\ .
\end{aligned}$$

The first term on the right can be dealt with in the same way as before for F_N:

$$\int_{\mathbb{I}^d} \varphi_\varepsilon(s-t)\,\tilde{x}(t)dt = \int_{\mathbb{R}^d} \varphi_\varepsilon(s-t)\mathbb{1}_{\mathbb{I}^d}(t)\,\tilde{x}(t)dt.$$

Then by Theorem 3.3.3 for any $\alpha > 0$ there exists a projection $\tilde{e}_1 \in \mathcal{M}_\theta$ such that

$$\nu(\tilde{e}_1^\perp) \lesssim \frac{\|x\|_1}{\alpha} \quad \text{and} \quad \big\|\tilde{e}_1\big[\int_{\mathbb{I}^d} \varphi_\varepsilon(\cdot - t)\,\tilde{x}(t)dt\big]\tilde{e}_1\big\|_\infty \leq \alpha, \quad \forall\, \varepsilon > 0.$$

On the other hand, for $s, t \in \mathbb{I}^d$ and $m \neq \mathbf{0}$ we have

$$\varphi_\varepsilon(s-t+m) \lesssim \frac{1}{\varepsilon^d}\,(1 + \frac{|m|}{\varepsilon})^{-d-\delta}.$$

Note that

$$\sum_{m \neq 0} \frac{1}{\varepsilon^d}\,(1+\frac{|m|}{\varepsilon})^{-d-\delta} \approx \frac{1}{\varepsilon^d} \sum_{1 \leq |m| \leq \varepsilon} + \varepsilon^\delta \sum_{\varepsilon < |m|} \frac{1}{|m|^{d+\delta}} \lesssim 1.$$

Hence (recalling that $x \geq 0$),

$$\sum_{m \neq 0} \int_{\mathbb{I}^d} \varphi_\varepsilon(s-t+m)\,\tilde{x}(t)dt \lesssim \sum_{m \neq 0} \frac{1}{\varepsilon^d}\,(1+\frac{|m|}{\varepsilon})^{-d-\delta} \int_{\mathbb{I}^d} \tilde{x}(t)dt \lesssim \int_{\mathbb{I}^d} \tilde{x}(t)dt.$$

The last integral is an operator in $L_1(\widetilde{\mathbb{T}_\theta^d})$ and its L_1-norm is less than or equal to that of x. Thus there exists a projection $\tilde{e}_2 \in \widetilde{\mathbb{T}_\theta^d}$ such that

$$\nu(\tilde{e}_2^\perp) \lesssim \frac{\|x\|_1}{\alpha} \quad \text{and} \quad \big\|\tilde{e}_2\big[\int_{\mathbb{I}^d} \tilde{x}(t)dt\big]\tilde{e}_2\big\|_\infty \leq \alpha.$$

Let $\tilde{e} = \tilde{e}_1 \vee \tilde{e}_2$. Then $\tilde{e}$ is a projection in $\mathcal{M}_\theta$, and combining the preceding two parts we get

$$\nu(\tilde{e}^\perp) \lesssim \frac{\|x\|_1}{\alpha} \quad \text{and} \quad \big\|\tilde{e}\,\widetilde{\Phi^\varepsilon[x]}\,\tilde{e}\big\|_\infty \leq \alpha, \quad \forall\, \varepsilon > 0.$$

We then deduce the weak type $(1,1)$ inequality for Φ^ε thanks to Lemma 3.3.5. The type (p,p) inequality is proved similarly. Therefore, the proof of Theorem 3.3.2 is complete. □

3.4 Pointwise convergence

In this section we apply the maximal inequalities proved in the previous section to study the pointwise convergence of Fourier series on quantum tori. To this end we first need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the almost uniform convergence introduced by Lance [48].

Let $(x_\lambda)_{\lambda\in\Lambda}$ be a family of elements in $L_p(\mathcal{M})$. Recall that $(x_\lambda)_{\lambda\in\Lambda}$ is said to converge almost uniformly to x, abbreviated as $x_\lambda \xrightarrow{a.u} x$, if for every $\epsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that

$$\tau(1-e) < \epsilon \quad \text{and} \quad \lim_\lambda \|(x_\lambda - x)e\|_\infty = 0.$$

Also, $(x_\lambda)_{\lambda\in\Lambda}$ is said to converge bilaterally almost uniformly to x, abbreviated as $x_\lambda \xrightarrow{b.a.u} x$, if the limit above is replaced by

$$\lim_\lambda \|e(x_\lambda - x)e\|_\infty = 0.$$

In the commutative case, both convergences are equivalent to the usual almost everywhere convergence thanks to Egorov's theorem. However, they are different in the noncommutative setting.

Theorem 3.4.1. *Let $1 \le p \le \infty$ and $x \in L_p(\mathbb{T}^d_\theta)$. Then $F_N[x] \xrightarrow{b.a.u.} x$ as $N \to \infty$. Moreover, for $2 \le p \le \infty$ the b.a.u. convergence can be strengthened to a.u. convergence.*

Similar statements hold for the two Poisson means P_r, $\mathbb{P}_r$ as $r \to \infty$ as well as for the mean Φ^ε as $\varepsilon \to 0$.

Proof. Let $x \in L_1(\mathbb{T}^d_\theta)$ and $\epsilon > 0$. Let (ε_m) and (δ_m) be two sequences of small positive numbers. Then for each $m \ge 1$ choose $y_m \in \mathcal{A}_\theta$ such that $\|x - y_m\|_1 \le \delta_m$. Let $z_m = x - y_m$, so $x = y_m + z_m$. Applying Theorem 3.3.2 to each z_m, we find a projection e_m such that

$$\sup_N \|e_m F_N[z_m] e_m\|_\infty \le \varepsilon_m \quad \text{and} \quad \tau(e_m^\perp) \le C\|z_m\|_1 \varepsilon_m^{-1} \le C\delta_m \varepsilon_m^{-1}.$$

The first inequality implies that

$$\|e_m z_m e_m\|_\infty \le \varepsilon_m.$$

Let $e = \bigwedge_m e_m$. Then

$$\tau(e^\perp) \le C \sum_m \delta_m \varepsilon_m^{-1} < \epsilon$$

provided ε_m and δ_m are appropriately chosen. On the other hand,

$$\begin{aligned}\|e(F_N[x] - x)e\|_\infty &\le \|e(F_N[y_m] - y_m)e\|_\infty + \|eF_N[z_m]e\|_\infty + \|ez_m e\|_\infty \\ &\le \|F_N[y_m] - y_m\|_\infty + 2\varepsilon_m.\end{aligned}$$

By Proposition 3.2.1,

$$\lim_{N\to\infty} \|F_N[y_m] - y_m\|_\infty = 0$$

for $y_m \in \mathcal{A}_\theta$. It then follows that

$$\limsup_{N\to\infty} \|e(F_N[x] - x)e\|_\infty \le 2\varepsilon_m.$$

Whence $\lim_{N\to\infty}\|e(F_N[x]-x)e\|_\infty = 0$. Therefore, $F_N[x]$ converges to x b.a.u. The b.a.u. convergence statements for the other summation methods are proved exactly in the same way.

Let us turn to the a.u. convergence. Let $x \in L_2(\mathbb{T}^d_\theta)$ and $\epsilon > 0$. We can assume x selfadjoint. As in the preceding argument, let $x = y_m + z_m$ with $y_m \in \mathcal{A}_\theta$ and $\|z_m\|_2 \le \delta_m$. Both y_m and z_m can be chosen selfadjoint. Now applying Theorem 3.3.2 to y_m^2, we find a projection e_m such that

$$\sup_N \|e_m F_N[z_m^2]e_m\|_\infty \le \varepsilon_m \quad \text{and} \quad \tau(e_m^\perp) \le C\varepsilon_m^{-1}\tau(z_m^2) \le C\varepsilon_m^{-1}\delta_m^2.$$

Since the map $z \mapsto F_N[z]$ is positive, by Kadison's Cauchy-Schwarz inequality [47], we have

$$\big(F_N[z_m]\big)^2 \le F_N[z_m^2].$$

Thus

$$\|F_N[z_m]e_m\|_\infty^2 \le \|e_m F_N[z_m^2]e_m\|_\infty \le \varepsilon_m. \tag{3.4.1}$$

Let $e = \bigwedge_m e_m$. Then $\tau(e^\perp) \le \epsilon$ for appropriate ε_m and δ_m and $\lim_N \|(F_N[x]-x)e\|_\infty = 0$. Therefore, $F_N[x] \xrightarrow{a.u.} x$. The proof of the corresponding statements for P_r and $\mathbb{P}_r$ is the same.

However, a minor extra argument is required for the mean Φ^ε because the map $z \mapsto \Phi^\varepsilon[z]$ is not positive in general. So we cannot apply directly Kadison's inequality to this map. But what is really missing is the one-sided weak type $(1,1)$ maximal inequality (3.4.1) for Φ^ε instead of F_N. In order to show this latter inequality, we can assume, as in the proof of Theorem 3.3.2, that φ is nonnegative. Then the kernel K_ε in (3.2.4) is nonnegative too. Thus the map $z \mapsto K_\varepsilon * \tilde{z}$ is positive, so we can apply Kadison's inequality to this map. Then as before for F_N, we get the desired inequality (3.4.1) with F_N replaced by Φ^ε, and then deduce that $\Phi^\varepsilon[x] \xrightarrow{a.u.} x$ as $\varepsilon \to 0$. Therefore, the theorem is completely proved. □

3.5 Bochner-Riesz means

As pointed out in section 3.2, when $\alpha > (d-1)/2$, the function Φ and φ associated with the Bochner-Riesz mean satisfy (3.2.1). Therefore, by Proposition 3.2.1, Theorems 3.3.2 and 3.4.1, we get the following

Proposition 3.5.1. *Let $\alpha > (d-1)/2$ and $x \in L_p(\mathbb{T}^d_\theta)$ with $1 \le p \le \infty$. Then*

i) $\lim\limits_{R\to\infty} B_R^\alpha[x] = x$ *in $L_p(\mathbb{T}^d_\theta)$ (relative to the w^*-topology for $p = \infty$).*

ii) $\big\|{\sup_{R>0}}^+ B_R^\alpha[x]\big\|_p \lesssim \|x\|_p$ *for $p > 1$.*

iii) $B_R^\alpha[x] \xrightarrow{b.a.u} x$ *as $R \to \infty$.*

If α is below the critical index $(d-1)/2$, the above results usually fail even in the scalar case, see for example [83, VII.4]. However, we have the following theorem, i.e., Theorem 3.5.2, which is the noncommutative analogue of Stein's theorem [81] (see also [83, VII.5]).

Theorem 3.5.2. *Let $1 < p < \infty$ and $\alpha > (d-1)|\frac{1}{2} - \frac{1}{p}|$. Then for any $x \in L_p(\mathbb{T}^d_\theta)$*

i) $\big\|{\sup_{R>0}}^+ B_R^\alpha[x]\big\|_p \lesssim \|x\|_p$ *with the relevant constant depends only p, d and α.*

ii) $\lim_{R\to\infty} B_R^\alpha[x] = x$ *in* $L_p(\mathbb{T}_\theta^d)$.

iii) $B_R^\alpha[x] \xrightarrow{b.a.u} x$ *as* $R \to \infty$.

Proof. The hard part of the theorem is the maximal inequality i). Assuming this part, it is easy to show the two others. Indeed, i) implies that for any $R > 0$

$$\big\|B_R^\alpha[x]\big\|_p \le \big\|{\sup_{r>0}}^+ B_r^\alpha[x]\big\|_p \lesssim \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}_\theta^d).$$

Whence

$$\sup_{R>0} \big\|B_R^\alpha\big\|_{L_p\to L_p} < \infty.$$

Together with the density of polynomials in $L_p(\mathbb{T}_\theta^d)$, this implies the mean convergence in ii). The pointwise convergence iii) can be proved as Theorem 3.4.1. The only thing to note is the fact that the type (p,p) maximal inequality in i) implies the corresponding weak type (p,p) inequality. The details are left to the reader.

The remainder of this section is devoted to the proof of i). We will follow the patten set up by Stein in the classical setting. The proof is quite technical and complicated, but essentially everything is based on two main ideas: estimate maximal function and square function by duality and interpolation.

We will frequently use the duality between $L_{p'}(\mathbb{T}_\theta^d;\ell_1)$ and $L_p(\mathbb{T}_\theta^d;\ell_\infty)$ (p' being the conjugate index of p). For the convenience of the reader we recall this duality. $L_{p'}(\mathbb{T}_\theta^d;\ell_1)$ is defined to be the space of all sequences $y = (y_n)$ in $L_{p'}(\mathbb{T}_\theta^d)$ which can be decomposed as

$$y_n = \sum_{k\ge1} u_{kn}^* v_{kn}, \quad \forall n \ge 1$$

for two families $(u_{kn})_{k,n\ge1}$ and $(v_{kn})_{k,n\ge1}$ in $L_{2p'}((\mathbb{T}_\theta^d)$ such that

$$\sum_{k,n\ge1} u_{kn}^* u_{kn} \in L_{p'}(\mathbb{T}_\theta^d) \ \text{ and } \ \sum_{k,n\ge1} v_{kn}^* v_{kn} \in L_{p'}(\mathbb{T}_\theta^d).$$

$L_{p'}(\mathbb{T}_\theta^d;\ell_1)$ is equipped with the norm

$$\|y\|_{L_{p'}(\mathbb{T}_\theta^d;\ell_1)} = \inf \big\|\sum_{k,n\ge1} u_{kn}^* u_{kn}\big\|_{p'}^{1/2} \big\|\sum_{k,n\ge1} v_{kn}^* v_{kn}\big\|_{p'}^{1/2},$$

where the infimum runs over all decompositions of y as above. It is easy to see that if $y_n \ge 0$ for all n, then $(y_n) \in L_{p'}(\mathbb{T}_\theta^d;\ell_1)$ iff $\sum_n y_n \in L_{p'}(\mathbb{T}_\theta^d)$. In this case, we have

$$\|y\|_{L_{p'}(\mathbb{T}_\theta^d;\ell_1)} = \big\|\sum_n y_n\big\|_{p'}.$$

Let $1 \le p' < \infty$. Then the dual space of $L_{p'}(\mathbb{T}_\theta^d;\ell_1)$ is $L_p(\mathbb{T}_\theta^d;\ell_\infty)$. The duality bracket is given by

$$\langle x,\ y\rangle = \sum_n \tau(x_n y_n), \quad x = (x_n) \in L_p(\mathbb{T}_\theta^d;\ell_\infty),\ y = (y_n) \in L_{p'}(\mathbb{T}_\theta^d;\ell_1).$$

We refer to [35] and [45] for more information.

For clarity we divide the proof of i) into three steps.

Step 1. If $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > \frac{d-1}{2}$, then for $1 < p \leq \infty$,

$$\big\|{\sup_{R>0}}^+ B_R^\alpha[x]\big\|_p \lesssim \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}_\theta^d).$$

To this end, choose $\delta > 0$ and $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > \delta > \frac{d-1}{2}$ and $\alpha = \delta + \beta$. We have the following identity

$$B_R^\alpha = C_{\beta,\delta} R^{-2\alpha} \int_0^R (R^2 - t^2)^{\beta-1} t^{2\delta+1} B_t^\delta dt, \tag{3.5.1}$$

where $C_{\beta,\delta} = 2\Gamma(\beta+\delta+1)/[\Gamma(\delta+1)\Gamma(\beta)]$. Let (R_n) be a sequence in $(0,\infty)$ and (y_n) an element in the unit ball of $L_{p'}(\mathbb{T}_\theta^d;\ell_1)$. Then, for any $x \in L_p(\mathbb{T}_\theta^d)$ we have

$$\begin{aligned}
\Big|\tau\Big(\sum_n B_{R_n}^\alpha[x]y_n\Big)\Big| &= |C_{\beta,\delta}|\Big|\sum_n R_n^{-2\alpha}\int_0^{R_n}(R_n^2-t^2)^{\beta-1}t^{2\delta+1}\tau(B_t^\delta[x]y_n)dt\Big| \\
&\leq |C_{\beta,\delta}|\int_0^1 |(1-t^2)^{\beta-1}t^{2\delta+1}|\Big|\tau\Big(\sum_n B_{tR_n}^\delta[x]y_n\Big)\Big|dt \\
&\leq |C_{\beta,\delta}|\int_0^1 |(1-t^2)^{\beta-1}t^{2\delta+1}|dt\big\|{\sup}_{R>0}^+ B_R^\delta[x]\big\|_p \\
&\lesssim \|x\|_p,
\end{aligned}$$

where we have used Proposition 3.5.1 ii) in the last inequality and the fact that

$$\int_0^1 |(1-t^2)^{\beta-1}t^{2\delta+1}|dt = \int_0^1 (1-t^2)^{\mathrm{Re}(\beta)-1}t^{2\delta+1}dt < \infty$$

since $\mathrm{Re}(\beta) = \mathrm{Re}(\alpha) - \delta > 0$ and $\delta > 0$. By duality we then deduce the desired maximal inequality.

Step 2. If $\alpha > 0$, then

$$\big\|{\sup_{R>0}}^+ B_R^\alpha[x]\big\|_2 \lesssim \|x\|_2, \quad \forall\, x \in L_2(\mathbb{T}_\theta^d). \tag{3.5.2}$$

We first consider the case of $\alpha > 1/2$. Choose $\beta > 1$ such that $\alpha = \beta + \delta$ with $\delta > -1/2$. By (3.5.1)

$$\begin{aligned}
B_R^{\beta+\delta} &= -C_{\beta,\delta}R^{-2(\beta+\delta)}\int_0^R\Big(\int_0^t B_r^\delta dr\Big)[(R^2-t^2)^{\beta-1}t^{2\delta+1}]'dt \\
&= C_{\beta,\delta}\int_0^1 \varphi(t)M_{Rt}^\delta dt,
\end{aligned}$$

where

$$M_t^\delta = \frac{1}{t}\int_0^t B_r^\delta dr \quad \text{and} \quad \varphi(t) = 2(\beta-1)(1-t^2)^{\beta-2}t^{2\delta+3} - (2\delta+1)(1-t^2)^{\beta-1}t^{2\delta+1}.$$

Note that $\int_0^1 |\varphi(t)|dt < \infty$. We will use the following fact that for any $(x_n) \in L_2(\mathbb{T}_\theta^d;\ell_\infty)$ one has

$$\big\|{\sup_n}^+ x_n\big\|_2 \approx \sup\Big\{\Big|\sum_n \tau(x_n y_n)\Big| \ :\ y_n \in L_2^+(\mathbb{T}_\theta^d),\ \Big\|\sum_n y_n\Big\|_2 \leq 1\Big\}$$

with universal equivalence constants (see [35, 45]). In what follows, we fix $x \in L_2(\mathbb{T}_\theta^d)$ and always assume that (R_n) is a sequence in $(0, \infty)$ and (y_n) a sequence of positive elements in $L_2(\mathbb{T}_\theta^d)$ with $\|\sum_n y_n\|_2 \leq 1$. Since

$$\begin{aligned}\Big|\tau\Big(\sum_n B_{R_n}^\alpha[x]y_n\Big)\Big| &= |C_{\beta,\delta}|\Big|\tau\Big(\sum_n\Big(\int_0^1 \varphi(t)M_{R_nt}^\delta(x)dt\Big)y_n\Big)\Big| \\ &\leq |C_{\beta,\delta}|\int_0^1 |\varphi(t)|\Big|\tau\Big(\sum_n M_{R_nt}^\delta(x)y_n\Big)\Big|dt \\ &\lesssim \|{\sup}_{R>0}^+ M_R^\delta(x)\|_2 \int_0^1 |\varphi(t)|dt,\end{aligned}$$

where we have used duality in the last inequality. We then deduce that

$$\big\|\sup_{R>0}{}^+ B_R^\alpha[x]\big\|_2 \lesssim \big\|\sup_{R>0}{}^+ M_R^\delta(x)\big\|_2.$$

Now we must show that

$$\big\|\sup_{R>0}{}^+ M_R^\delta(x)\big\|_2 \lesssim \|x\|_2 \quad \text{if} \quad \delta > -1/2. \tag{3.5.3}$$

To this end, we again use duality. We have

$$\begin{aligned}\Big|\tau\Big(\sum_n M_{R_n}^\delta(x)y_n\Big)\Big| &\leq \Big|\tau\Big(\sum_n M_{R_n}^{\delta+1}(x)y_n\Big)\Big| + \Big|\tau\Big(\sum_n [M_{R_n}^{\delta+1}(x) - M_{R_n}^\delta(x)]y_n\Big)\Big| \\ &\leq \big\|\sup_{R>0}{}^+ M_R^{\delta+1}(x)\big\|_2 + \Big|\tau\Big(\sum_n G_{R_n}^\delta(x)y_n\Big)\Big|,\end{aligned}$$

where $G_R^\delta(x) = M_R^{\delta+1}(x) - M_R^\delta(x)$. Using the following elementary inequality

$$|\tau(ab)|^2 \leq \tau(|a|b)\tau(|a^*|b), \quad \forall\, a, b \in \mathbb{T}_\theta^d \text{ with } b \geq 0,$$

we have

$$\Big|\tau\Big(\sum_n G_{R_n}^\delta(x)y_n\Big)\Big|^2 \leq \tau\Big(\sum_n |G_{R_n}^\delta(x)|y_n\Big)\tau\Big(\sum_n |G_{R_n}^\delta(x)^*|y_n\Big).$$

Note that

$$\begin{aligned}|G_R^\delta(x)| &= \Big|\frac{1}{R}\int_0^R [B_r^{\delta+1}[x] - B_r^\delta[x]]\,dr\Big| \\ &\leq \Big(\int_0^R |B_r^{\delta+1}[x] - B_r^\delta[x]|^2\frac{dr}{R}\Big)^{1/2} \leq G^\delta(x),\end{aligned}$$

where

$$G^\delta(x) = \Big(\int_0^\infty |B_r^{\delta+1}[x] - B_r^\delta[x]|^2\frac{dr}{r}\Big)^{1/2}.$$

It then follows that

$$\tau\big(\sum_n |G_{R_n}^\delta(x)|y_n\big) \leq \tau\big(G^\delta(x)\sum_n y_n\big) \leq \|G^\delta(x)\|_2\big\|\sum_n y_n\big\|_2 \leq \|G^\delta(x)\|_2.$$

Similarly,

$$\tau\big(\sum_n |G_{R_n}^\delta(x)^*|y_n\big) \leq \|G_*^\delta(x)\|_2,$$

where

$$G_*^\delta(x) = \Big(\int_0^\infty |(B_r^{\delta+1}[x] - B_r^\delta[x])^*|^2\frac{dr}{r}\Big)^{1/2}.$$

Combining the preceding inequalities, we obtain

$$\big\|{\sup_{R>0}}^+ M_R^\delta(x)\big\|_2 \leq \big\|{\sup_{R>0}}^+ M_R^{\delta+1}(x)\big\|_2 + \|G^\delta(x)\|_2^{1/2}\|G_*^\delta(x)\|_2^{1/2}.$$

We now claim that

$$\max\big(\|G^\delta(x)\|_2,\ \|G_*^\delta(x)\|_2\big) \lesssim \|x\|_2, \quad \text{if} \quad \delta > -1/2.$$

Indeed, by Parseval's identity we have

$$\begin{aligned}
\|G^\delta(x)\|_2^2 &= \int_0^\infty \tau\big(\big|B_r^{\delta+1}[x] - B_r^\delta[x]\big|^2\big)\frac{dr}{r}\\
&= \int_0^\infty \sum_{|m|_2\leq R}\Big|\Big(1-\frac{|m|_2^2}{r^2}\Big)^{\delta+1} - \Big(1-\frac{|m|_2^2}{r^2}\Big)^\delta\Big|^2|\hat{x}(m)|^2\frac{dr}{r}\\
&= \sum_{m\neq 0}|\hat{x}(m)|^2\int_{|m|_2}^\infty \frac{|m|_2^4}{r^4}\Big(1-\frac{|m|_2^2}{r^2}\Big)^\delta\frac{dr}{r}\\
&\lesssim \|x\|_2^2
\end{aligned}$$

because the integral

$$\int_{|m|_2}^\infty \frac{|m|_2^4}{r^4}\Big(1-\frac{|m|_2^2}{r^2}\Big)^\delta\frac{dr}{r} = \int_1^\infty r^{-5}(1-r^{-2})^{2\delta}dr < \infty$$

if $\delta > -1/2$. In the same way, we have

$$\|G_*^\delta(x)\|_2 \lesssim \|x\|_2.$$

Hence our claim is proved. Consequently,

$$\big\|{\sup_{R>0}}^+ M_R^\delta(x)\big\|_2 \lesssim \big\|{\sup_{R>0}}^+ M_R^{\delta+1}(x)\big\|_2 + \|x\|_2.$$

Then by iteration, for any positive integer k we have

$$\big\|{\sup_{R>0}}^+ M_R^\delta(x)\big\|_2 \lesssim \big\|{\sup_{R>0}}^+ M_R^{\delta+k}(x)\big\|_2 + \|x\|_2.$$

Now, if we choose k such that $\delta + k > (d-1)/2$, then using *Step 1*, we have

$$\big\|{\sup_{R>0}}^+ M_R^{\delta+k}[x]\big\|_2 \leq \big\|{\sup_{R>0}}^+ B_R^{\delta+k}[x]\big\|_2 \lesssim \|x\|_2.$$

Therefore, we deduce (3.5.3), and hence (3.5.2) provided $\alpha > 1/2$.

We now deal with the general case of $\alpha > 0$. Choose $\beta > 1/2$ and $\delta > -1/2$ so that $\alpha = \beta + \delta$. Then by (3.5.1)

$$\begin{aligned}
B_R^{\beta+\delta} - \frac{C_{\beta,\delta}}{C_{\beta,\delta+1}}B_R^{\beta+\delta+1} &= C_{\beta,\delta}R^{-2(\beta+\delta)}\Big[\int_0^R(R^2-t^2)^{\beta-1}t^{2\delta+1}B_t^\delta dt\\
&\qquad - R^{-2}\int_0^R(R^2-t^2)^{\beta-1}t^{2(\delta+1)+1}B_t^{\delta+1}dt\Big]\\
&= C_{\beta,\delta}R^{-2(\beta+\delta)}\Big[\int_0^R(R^2-t^2)^{\beta-1}t^{2\delta+1}(B_t^\delta - B_t^{\delta+1})dt\\
&\qquad + \int_0^R(R^2-t^2)^{\beta-1}t^{2\delta+1}(1-R^{-2}t^2)B_t^{\delta+1}dt\Big]\\
&\triangleq \mathrm{I}_R + \mathrm{II}_R.
\end{aligned}$$

We first estimate I_R. By the argument already used above

$$\big|\tau\big(\sum_n \mathrm{I}_{R_n}(x)y_n\big)\big|^2 \le \tau\big(\sum_n |\mathrm{I}_{R_n}(x)|y_n\big)\tau\big(\sum_n |\mathrm{I}_{R_n}(x)^*|y_n\big).$$

However,

$$\begin{aligned}
|\mathrm{I}_R(x)| &= |C_{\beta,\delta}|R^{-2(\beta+\delta)}\Big|\int_0^R (R^2-t^2)^{\beta-1}t^{2\delta+1}(B_t^{\delta+1}[x]-B_t^{\delta}[x])dt\Big|\\
&\le |C_{\beta,\delta}|R^{-2(\beta+\delta)}\Big(\int_0^R |(R^2-t^2)^{\beta-1}t^{2\delta+1}|^2 dt\Big)^{1/2}\\
&\quad\times R^{1/2}R^{-1/2}\Big(\int_0^R |B_t^{\delta+1}[x]-B_t^{\delta}[x]|^2 dt\Big)^{1/2}\\
&\lesssim G^{\delta}(x)
\end{aligned}$$

because the integral

$$R^{1-4(\beta+\delta)}\int_0^R |(R^2-t^2)^{\beta-1}t^{2\delta+1}|^2 dt = \int_0^1 |(1-t^2)^{\beta-1}t^{2\delta+1}|^2 dt < \infty$$

when $\beta > 1/2$. Similarly,

$$|\mathrm{I}_R(x)^*| \lesssim G_*^{\delta}(x).$$

Hence, we deduce

$$\big\|{\sup_{R>0}}^+ \mathrm{I}_R(x)\big\|_2 \lesssim \|G^{\delta}(x)\|_2^{1/2}\|G_*^{\delta}(x)\|_2^{1/2} \lesssim \|x\|_2.$$

Next, we estimate the second term II_R. Since

$$\begin{aligned}
\mathrm{II}_R &= C_{\beta,\delta}R^{-2(\beta+\delta)}\int_0^R (R^2-t^2)^{\beta-1}t^{2\delta+1}(1-R^{-2}t^2)B_t^{\delta+1}dt\\
&= C_{\beta,\delta}R^{-2(\beta+\delta)-2}\int_0^R (R^2-t^2)^{\beta}t^{2\delta+1}B_t^{\delta+1}dt
\end{aligned}$$

and $\beta > 1/2$, II_R can be dealt with as B_R^{α} in the case of $\alpha > 1/2$. So we conclude that

$$\big\|{\sup_{R>0}}^+ B_R[x]\big\|_2 \lesssim \|x\|_2.$$

Therefore, we have finally arrived at

$$\begin{aligned}
\big\|{\sup_{R>0}}^+ B_R^{\beta+\delta}(x)\big\|_2 &\le \frac{|C_{\beta,\delta}|}{|C_{\beta,\delta+1}|}\big\|{\sup_{R>0}}^+ B_R^{\beta+\delta+1}[x]\big\|_2\\
&\quad+ \big\|{\sup_{R>0}}^+ \mathrm{I}_R(x)\big\|_2 + \big\|{\sup_{R>0}}^+ \mathrm{II}_R[x]\big\|_2\\
&\lesssim \|x\|_2.
\end{aligned}$$

This completes the proof of *Step 2.*

Step 3. When p is near 1 or ∞, the announced result is in fact already contained in *Step 1.* Moreover, *Step 2* gives the desired inequality in the special case of $p = 2$. The general case can be deduced from these special ones by applying Stein's complex interpolation. To this end, we need first a strengthening of (3.5.2) which allows the order α to be complex, that is,

$$\big\|{\sup_{R>0}}^+ B_R^{\alpha}[x]\big\|_2 \lesssim \|x\|_2, \quad \alpha \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0. \tag{3.5.4}$$

This can be reduced to the case of $\alpha > 0$ by using the argument in *Step 1.* We omit the details.

Let $x \in L_p(\mathbb{T}_\theta^d)$ with $\|x\|_p < 1$ and $y = (y_n)$ be a finite sequence in $L_{p'}(\mathbb{T}_\theta^d)$ with $\|y\|_{L_{p'}(\mathbb{T}_\theta^d;\ell_1)} < 1$. Assume first that $p < 2$. For any fixed $\alpha > (d-1)(1/p - 1/2)$ we can always choose $p_1 > 1, \alpha_0 > 0$ and $\alpha_1 > (d-1)/2$ such that

$$\alpha = (1-t)\alpha_0 + t\alpha_1 \quad \text{and} \quad \frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1}$$

for some $0 < t < 1$. Define

$$f(z) = u|x|^{\frac{p(1-z)}{2} + \frac{pz}{p_1}}, \quad z \in \mathbb{C},$$

where $x = u|x|$ is the polar decomposition of x. On the other hand, by Proposition 2.5 of [45], there is a function $g = (g_n)_n$ continuous on the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and analytic in the interior such that $g(t) = y$ and

$$\sup_{s\in\mathbb{R}} \max\big\{\|g(\mathrm{i}s)\|_{L_2(\mathbb{T}_\theta^d;\ell_1)}, \|g(1+\mathrm{i}s)\|_{L_{p_1'}(\mathbb{T}_\theta^d;\ell_1)}\big\} < 1.$$

Fix a sequence $(R_n) \subset (0, \infty)$ and $\delta > 0$. We define

$$F(z) = \exp\big(\delta(z^2 - t^2)\big) \sum_n \tau\big(B_{R_n}^{(1-z)\alpha_0 + z\alpha_1}[f(z)]g_n(z)\big).$$

F is a function analytic in the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. By (3.5.4), for any $s \in \mathbb{R}$ we have

$$\begin{aligned} |F(\mathrm{i}s)| &\le \exp\big(-\delta(s^2 + t^2)\big)\big\|\big(B_{R_n}^{\alpha_0 + \mathrm{i}s(\alpha_1-\alpha_0)}(f(\mathrm{i}s))\big)_n\big\|_{L_2(\mathbb{T}_\theta^d;\ell_\infty)}\big\|g(\mathrm{i}s)\big\|_{L_2(\mathbb{T}_\theta^d;\ell_1)} \\ &\lesssim \|f(\mathrm{i}s)\|_2 \lesssim 1. \end{aligned}$$

Similarly, by *Step 1* we have

$$|F(1+\mathrm{i}s)| \lesssim 1.$$

Therefore, by the maximum principle we get $|F(t)| \lesssim 1$ i.e.,

$$\big|\tau\big(\sum_n B_{R_n}^\alpha[x]y_n\big)\big| \lesssim 1$$

if $\|x\|_{L_p(\mathcal{N}_\theta)} < 1$. Then by duality and homogeneity, we deduce that

$$\big\|{\sup_{R>0}}^+ B_R^\alpha[x]\big\|_p \lesssim \|x\|_p, \quad \forall\, x \in L_p(\mathbb{T}_\theta^d).$$

The argument for the case of $p > 2$ is similar once we begin by setting $p_1 = \infty$. Thus the proof of Theorem 3.5.2 is complete. □

Remark 3.5.3. The previous proof gives a slightly more general result by allowing α to be complex. Namely, Theorem 3.5.2 remains true under the assumption that $\mathrm{Re}(\alpha) > (d-1)|\frac{1}{2} - \frac{1}{p}|$ with $\alpha \in \mathbb{C}$ and $1 < p < \infty$.

Remark 3.5.4. Let $\mathcal{M}$ be a semifinite von Neumann algebra. Then Theorem 3.5.2 admits the following analogue for the algebra $\mathbb{T}^d\overline{\otimes}\mathcal{M}$ with the same proof: Let $1 < p \le \infty$ and $\mathrm{Re}(\alpha) > (d-1)|\frac{1}{2} - \frac{1}{p}|$. Then

$$\big\|{\sup_{R>0}}^+ B_R^\alpha[f]\big\|_p \lesssim \|f\|_p, \quad \forall\, f \in L_p(\mathbb{T}^d; L_p(\mathcal{M})).$$

Moreover, $B_R^\alpha[f]$ converges b.a.u. to f as $R \to \infty$. Here

$$B_R^\alpha[f] = \sum_{|m|_2 \le R} \Big(1 - \frac{|m|_2^2}{R^2}\Big)^\alpha \hat{f}(m) z^m$$

for $f \in L_p(\mathbb{T}^d; L_p(\mathcal{M}))$ with Fourier series expansion

$$f \sim \sum_{m \in \mathbb{Z}^d} \hat{f}(m) z^m.$$

3.6 Fourier multipliers

It is our intention in this section to study Fourier multipliers in the quantum d-torus $\mathbb{T}_\theta^d$. We will compare (completely) bounded L_p Fourier multipliers with those in the usual d-torus $\mathbb{T}^d$. The right framework for this investigation is the category of operator spaces.

We now recall some standard operator space notions and refer the reader to [21] and [68] for more information. A (concrete) operator space is a closed subspace E of $\mathcal{B}(H)$ for some Hilbert space H. Then E inherits the matricial structure of $\mathcal{B}(H)$ via the embedding $\mathbb{M}_n(E) \subset \mathbb{M}_n(\mathcal{B}(H))$. More precisely, let $\mathbb{M}_n(E)$ denote the space of $n \times n$ matrices with entries in E, equipped with the norm induced by $\mathcal{B}(\ell_2^n(H))$. An abstract matricial norm characterization of operator spaces was given by Ruan. The morphisms in the category of operator spaces are completely bounded maps. Let H, K be two Hilbert spaces. Suppose that $E \subset \mathcal{B}(H)$ and $F \subset \mathcal{B}(K)$ are two operator spaces. A map $u: E \to F$ is called completely bounded (in short c.b.) if

$$\sup_n \|\mathrm{id}_{\mathbb{M}_n} \otimes u\|_{\mathbb{M}_n(E) \to \mathbb{M}_n(F)} < \infty,$$

and the c.b. norm $\|u\|_{\mathrm{cb}}$ is defined to be the above supremum. We denote by $\mathrm{CB}(E, F)$ the space of all c.b. maps from E to F, equipped with the norm $\| \ \|_{\mathrm{cb}}$. This is a Banach space.

For an operator space E there exists a natural matricial structure on the Banach dual E^* of E so that E^* becomes an operator space too. The norm of $\mathbb{M}_n(E^*)$ is that of $\mathrm{CB}(E, \mathbb{M}_n)$ ($\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$). This is usually called the standard dual of E. We will simply say the dual of $\mathbb{E}$ since only standard duals are used in the sequel.

We will need the natural operator space structure on noncommutative L_p-spaces introduced by Pisier. Let $\mathcal{M}$ be a (semifinite) von Neumann algebra on a Hilbert space H. Then the embedding $\mathcal{M} \subset \mathcal{B}(H)$ gives to $\mathcal{M}$ an operator space structure. To equip $L_1(\mathcal{M})$ with an operator space structure, we view $L_1(\mathcal{M})$ as the predual of the opposite algebra $\mathcal{M}^{\mathrm{op}}$ instead of $\mathcal{M}$ itself. In this way, $L_1(\mathcal{M})$ becomes a subspace of the dual operator space of $\mathcal{M}^{\mathrm{op}}$. This is the natural operator space structure of $L_1(\mathcal{M})$. Then for any $1 < p < \infty$ the operator space structure of $L_p(\mathcal{M})$ is defined via the complex interpolation formula $L_p(\mathcal{M}) = \big(L_\infty(\mathcal{M}),\, L_1(\mathcal{M})\big)_{1/p}$. We refer the reader to [65, 68] for more details.

We will use the following fundamental property of c.b. maps between two noncommutative L_p-spaces due to Pisier [65]. Let $\mathcal{N}$ be another (semifinite) von Nuemann algebra. Then a map $u : L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is c.b. iff $\mathrm{id}_{S_p} \otimes u : L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M}) \to L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{N})$ is bounded. In this case,

$$\|u\|_{\mathrm{cb}} = \big\|\mathrm{id}_{S_p} \otimes u : L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M}) \to L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{N})\big\|.$$

Here S_p denotes the Schatten p-class, namely, the noncommutative L_p-space associated to $\mathcal{B}(\ell_2)$ equipped with the usual trace. The readers who are not very familiar with operator space theory can take this property as the definition of c.b. maps between noncommutative L_p-spaces.

Now we turn to Fourier multipliers on quantum tori. Let $\phi = (\phi_m)_{m\in\mathbb{Z}^d} \subset \mathbb{C}$. We define T_ϕ by

$$\widehat{T_\phi x}(m) = \phi_m \hat{x}(m),\ \forall m \in \mathbb{Z}^d,$$

for any polynomial $x \in \mathcal{P}_\theta$. We call ϕ a bounded L_p multiplier (resp. c.b. L_p multiplier) on the quantum torus $\mathbb{T}^d_\theta$ if T_ϕ extends to a bounded (resp. c.b.) map on $L_p(\mathbb{T}^d_\theta)$. The space of all L_p multipliers (resp. c.b. L_p multipliers) on $\mathbb{T}^d_\theta$ is denoted by $\mathrm{M}(L_p(\mathbb{T}^d_\theta))$ (resp. $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$), equipped with the natural norm (resp. c.b. norm). When $\theta = 0$, we recover the Fourier multipliers on the usual d-torus $\mathbb{T}^d$. The corresponding multiplier spaces are denoted by $\mathrm{M}(L_p(\mathbb{T}^d))$ and $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d))$, respectively.

The following remark summarizes some easily checked basic properties of quantum Fourier multipliers. We only state them for c.b. case, although all of them are equally valid for bounded multipliers.

Remark 3.6.1. Let $1 \le p, p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

i) $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$ is a Banach algebra under pointwise multiplication.

ii) $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta)) = \mathrm{M}_{\mathrm{cb}}(L_{p'}(\mathbb{T}^d_\theta))$.

iii) $\mathrm{M}_{\mathrm{cb}}(L_q(\mathbb{T}^d_\theta)) \subset \mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$, a contractive inclusion for $2 \le p \le q \le \infty$.

iv) $\mathrm{M}_{\mathrm{cb}}(L_2(\mathbb{T}^d_\theta)) = \mathrm{M}(L_2(\mathbb{T}^d_\theta)) = \ell_\infty(\mathbb{Z}^d)$ with equal norms.

It is well-known that in the classical case Fourier multipliers are closely related to Schur multipliers. We will exploit such a relation in the quantum case too. To this end we first recall the definition of Schur multipliers. Let Λ be an index set. The elements of $\mathcal{B}(\ell_2(\Lambda))$ are represented by infinite matrices in the canonical basis of $\ell_2(\Lambda)$. A complex function $\psi = (\psi_{st})$ on $\Lambda \times \Lambda$ (or matrix indexed by Λ) is called a bounded Schur multiplier on $\mathcal{B}(\ell_2(\Lambda))$ if for every operator $a = (a_{st}) \in \mathcal{B}(\ell_2(\Lambda))$, the matrix $(\psi_{st}a_{st})$ represents a bounded operator on $\ell_2(\Lambda)$. We then denote $M_\psi a = (\psi_{st}a_{st})$. In this case, M_ψ is necessarily bounded on $\mathcal{B}(\ell_2(\Lambda))$. More generally, for $1 \le p \le \infty$, if M_ψ induces a bounded map on the Schatten p-class $S_p(\ell_2(\Lambda))$ based on $\ell_2(\Lambda)$, we call ψ a bounded Schur multiplier on $S_p(\ell_2(\Lambda))$. Similarly, we define the completely boundedness of M_ψ.

Fourier and Schur multipliers are linked together via Toeplitz matrices. As usual, we represent $\mathbb{T}^d_\theta$ as a von Neumann algebra on $L_2(\mathbb{T}^d_\theta)$ by left multiplication. For every $x \in \mathbb{T}^d_\theta$, let $[x]$ denote the representation matrix of x on $\ell_2(\mathbb{Z}^d)$ in the orthonormal basis $(U^m)_{m\in\mathbb{Z}^d}$. Namely,

$$[x] = \big(\langle xU^n,\ U^m\rangle\big)_{m,n\in\mathbb{Z}^d}.$$

Let $\tilde{\theta}$ be the following $d \times d$-matrix deduced from the skew symmetric matrix θ:

$$\tilde{\theta} = -2\pi \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \dots & \theta_{1d} \\ 0 & 0 & \theta_{23} & \dots & \theta_{2d} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \theta_{d-1,d} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then by the commutation relation (3.1.1), we have

$$xU^n = \sum_k \hat{x}(k)U^kU^n = \sum_k \hat{x}(k)U_1^{k_1}\cdots U_d^{k_d}U_1^{n_1}\cdots U_d^{n_d} = \sum_k \hat{x}(k)e^{\mathrm{i}n\tilde{\theta}k^t}U^{k+n},$$

where $n=(n_1,\dots,n_d)$, k^t is the transpose of $k=(k_1,\dots,k_d)$ and $n\tilde{\theta}k^t$ denotes the matrix product. Thus

$$[x] = \Big(\hat{x}(m-n)e^{\mathrm{i}n\tilde{\theta}(m-n)^t}\Big)_{m,n\in\mathbb{Z}^d}. \tag{3.6.1}$$

If $\theta=0$, $[x]$ is a Toeplitz matrix. In the general case, $[x]$ is a twisted Toeplitz matrix.

For $\phi=(\phi_m)_{m\in\mathbb{Z}^d}\in\ell_\infty(\mathbb{Z}^d)$, we have

$$[T_\phi x] = (\phi_{m-n}\hat{x}(m-n)e^{\mathrm{i}n\tilde{\theta}(m-n)^t})_{m,n\in\mathbb{Z}^d} = M_{\tilde{\phi}}([x]), \tag{3.6.2}$$

where $\tilde{\phi}_{mn}=\phi_{m-n}$. This is the link between the Fourier and Schur multipliers associated to ϕ. This link remains valid for operators x in $\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d$. In this case, the entries of the twisted Toeplitz matrix $[x]$ are operators in $\mathcal{B}(\ell_2)$.

To illustrate the usefulness of the relationship above, let us show the following simple result.

Proposition 3.6.2. *We have*

$$\mathrm{M_{cb}}(\mathbb{T}_\theta^d) = \mathrm{M_{cb}}(L_\infty(\mathbb{T}^d)) = \mathrm{M}(L_\infty(\mathbb{T}^d)) \quad \textit{with equal norms.}$$

Proof. The argument below is standard. Let Γ_∞ denote the subspace of $\mathcal{B}(\ell_2(\mathbb{Z}^d))$ consisting of all twisted Toeplitz matrices of the form (3.6.1). By the preceding discussion, for any $x\in\mathbb{T}_\theta^d$ we have

$$\|T_\phi(x)\|_\infty = \|T_\phi(x)\|_{\mathcal{B}(L_2(\mathbb{T}_\theta^d))} = \|[T_\phi(x)]\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))} = \|M_{\tilde{\phi}}[x]\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))}.$$

Consequently,

$$T_\phi \text{ is bounded on } \mathbb{T}_\theta^d \iff M_{\tilde{\phi}}\big|_{\Gamma_\infty}:\Gamma_\infty\to\Gamma_\infty \text{ is bounded.}$$

Moreover, in this case,

$$\|T_\phi\| = \|M_{\tilde{\phi}}\big|_{\Gamma_\infty}\|.$$

Considering the vector-valued case where $x\in\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d$, we get the c.b. analogue of the above equivalence:

$$T_\phi \text{ is c.b. on } \mathbb{T}_\theta^d \iff M_{\tilde{\phi}}\big|_{\Gamma_\infty} \text{ is c.b. on } \Gamma_\infty \quad\text{and}\quad \|T_\phi\|_{\mathrm{cb}} = \big\|M_{\tilde{\phi}}\big|_{\Gamma_\infty}\big\|_{\mathrm{cb}}.$$

Thus, if $M_{\tilde{\phi}}$ is c.b. on $\mathcal{B}(\ell_2(\mathbb{Z}^d))$, then $M_{\tilde{\phi}}\big|_{\Gamma_\infty}$ is c.b. on Γ_∞, so is T_ϕ on $\mathbb{T}_\theta^d$.

Conversely, suppose $\phi\in\mathrm{M_{cb}}(\mathbb{T}_\theta^d)$. Let $V=\mathrm{diag}(\cdots,U^n,\cdots)$ be the diagonal matrix with diagonal entries $(U^n)_{n\in\mathbb{Z}^d}$. V is a unitary operator in $\mathcal{B}(\ell_2(\mathbb{Z}^d))\overline{\otimes}\mathbb{T}_\theta^d$. For any $a=(a_{mn})_{m,n\in\mathbb{Z}^d}\in\mathcal{B}(\ell_2(\mathbb{Z}^d))$, let $x=V(a\otimes 1_{\mathbb{T}_\theta^d})V^*\in\mathcal{B}(\ell_2(\mathbb{Z}^d))\overline{\otimes}\mathbb{T}_\theta^d$, where $1_{\mathbb{T}_\theta^d}$ denotes the unit of $\mathbb{T}_\theta^d$. Then

$$x = (U^m a_{mn}U^{-n})_{m,n\in\mathbb{Z}^d} = \sum_{m,n} a_{mn}e_{mn}\otimes U^mU^{-n} = \sum_{m,n} a_{mn}e_{mn}\otimes e^{-\mathrm{i}n\tilde{\theta}m^t}U^{m-n},$$

where (e_{mn}) are the canonical matrix units of $\mathcal{B}(\ell_2(\mathbb{Z}^d))$. Since V is unitary, we have

$$\|x\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))\overline{\otimes}\mathbb{T}_\theta^d} = \|a\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))}.$$

On the other hand,

$$(\mathrm{id}_{\mathcal{B}(\ell_2(\mathbb{Z}^d))} \otimes T_\phi)(x) = \sum_{m,n} \phi_{m-n} a_{mn} e_{mn} \otimes e^{-in\tilde{\theta}m^t} U^{m-n} = V(M_{\tilde{\phi}}(a) \otimes 1_{\mathbb{T}^d_\theta})V^*.$$

It then follows that

$$\begin{aligned}\|M_{\tilde{\phi}}(a)\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))} &= \|(\mathrm{id}_{\mathcal{B}(\ell_2(\mathbb{Z}^d))} \otimes T_\phi)(x)\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))\overline{\otimes}\mathbb{T}^d_\theta} \\ &\le \|T_\phi\|_{\mathrm{cb}} \|x\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))\overline{\otimes}\mathbb{T}^d_\theta} = \|T_\phi\|_{\mathrm{cb}} \|a\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))}.\end{aligned}$$

Therefore, $\tilde{\phi}$ is a bounded Schur multiplier on $\mathcal{B}(\ell_2(\mathbb{Z}^d))$. Considering matrices $a = (a_{mn})_{m,n\in\mathbb{Z}^d}$ with entries in $\mathcal{B}(\ell_2)$, i.e., $a = (a_{mn})_{m,n\in\mathbb{Z}^d} \in \mathcal{B}(\ell_2)\overline{\otimes}\mathcal{B}(\ell_2(\mathbb{Z}^d))$, we show in the same way that $M_{\tilde{\phi}}$ is c.b. on $\mathcal{B}(\ell_2(\mathbb{Z}^d))$, so $\tilde{\phi}$ is a c.b. Schur multiplier on $\mathcal{B}(\ell_2(\mathbb{Z}^d))$ and $\|M_{\tilde{\phi}}\|_{\mathrm{cb}} \le \|T_\phi\|_{\mathrm{cb}}$.

In summary, we have proved that

$$T_\phi \text{ is c.b. on } \mathbb{T}^d_\theta \iff M_{\tilde{\phi}} \text{ is c.b. on } \mathcal{B}(\ell_2(\mathbb{Z}^d)).$$

Applying this result to the commutative case ($\theta = 0$), we get that

$$T_\phi \text{ is c.b. on } L_\infty(\mathbb{T}^d) \iff M_{\tilde{\phi}} \text{ is c.b. on } \mathcal{B}(\ell_2(\mathbb{Z}^d)).$$

Therefore,

$$\mathrm{M_{cb}}(\mathbb{T}^d_\theta) = \mathrm{M_{cb}}(L_\infty(\mathbb{T}^d)) \quad \text{with equal norms.}$$

However, it is well known that a Fourier multiplier ϕ is bounded on $L_\infty(\mathbb{T}^d)$ iff it is the Fourier transform of a bounded Borel measure μ on $\mathbb{T}^d$. In this case, T_ϕ is the convolution operator by μ and its norm is equal to $\|\mu\|$. Then it is easy to check that T_ϕ c.b. on $L_\infty(\mathbb{T}^d)$. Thus

$$\mathrm{M_{cb}}(L_\infty(\mathbb{T}^d)) = \mathrm{M}(L_\infty(\mathbb{T}^d)) \quad \text{with equal norms.} \tag{3.6.3}$$

Combining the preceding results, we deduce the announced assertion. □

The main result of this section is the following theorem, which extends the first equality in the previous proposition to all $1 \le p \le \infty$. We point out that the inclusion $\mathrm{M_{cb}}(L_p(\mathbb{T}^d)) \subset \mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta))$ was proved independently by Junge, Mei and Parcet [39].

Theorem 3.6.3. *Let* $1 < p < \infty$. *Then* $\mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta)) = \mathrm{M_{cb}}(L_p(\mathbb{T}^d))$ *with equal norms.*

Proof. The inclusion $\mathrm{M_{cb}}(L_p(\mathbb{T}^d)) \subset \mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta))$ can be easily proved by transference. Indeed, let $\phi \in \mathrm{M_{cb}}(L_p(\mathbb{T}^d))$, and let $x \in L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)$ be a polynomial in U:

$$x = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) \otimes U^m,$$

where only a finite number of coefficients $\hat{x}(m)$ are nonzero operators in S_p. Let

$$\tilde{x}(z) = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) \otimes U^m z^m, \quad z \in \mathbb{T}^d.$$

Then $\tilde{x} \in L_p(\mathbb{T}^d; L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta))$ and

$$T_\phi(\tilde{x}) = \widetilde{T_\phi(x)},$$

where the first T_ϕ is viewed as a multiplier on $\mathbb{T}^d$ and the second on $\mathbb{T}^d_\theta$. Recall that $\mathbb{T}^d_\theta$ is hyperfinite, so the algebra $\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta$ can be approximated by matrix algebras. Therefore, the complete boundedness of $\mathbb{T}_\phi$ on $L_p(\mathbb{T}^d)$ implies

$$\|\widetilde{T_\phi(x)}\|_{L_p(\mathbb{T}^d;L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta))} \leq \|\phi\|_{\mathrm{M_{cb}}(L_p(\mathbb{T}^d))}\|\tilde{x}\|_{L_p(\mathbb{T}^d;L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta))}.$$

However, by Corollary 3.1.2

$$\|\widetilde{T_\phi(x)}\|_{L_p(\mathbb{T}^d;L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta))} = \|T_\phi(x)\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)}$$

and

$$\|\tilde{x}\|_{L_p(\mathbb{T}^d;L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta))} = \|x\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)}.$$

Thus

$$\|T_\phi(x)\|_p \leq \|\phi\|_{\mathrm{M_{cb}}(L_p(\mathbb{T}^d))}\|x\|_p.$$

Whence T_ϕ is c.b., so $\phi \in \mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta))$ and $\|\phi\|_{\mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta))} \leq \|\phi\|_{\mathrm{M_{cb}}(L_p(\mathbb{T}^d))}$.

For the converse inclusion, note that the argument in the second part of the proof of Proposition 3.6.2 works equally at the level of L_p-spaces. Thus we get that

$$T_\phi \text{ is c.b. on } L_p(\mathbb{T}^d_\theta) \implies M_{\tilde{\phi}} \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)).$$

Then using Neuwirth and Ricard's transference theorem [58], we deduce that T_ϕ is c.b. on $L_p(\mathbb{T}^d)$, so $\mathrm{M_{cb}}(L_p(\mathbb{T}^d_\theta)) \subset \mathrm{M_{cb}}(L_p(\mathbb{T}^d))$ contractively.

However, for reason of completeness, we include a self-contained proof in the spirit of the proof of Proposition 3.6.2 by adapting Neuwirth and Ricard's argument to the present setting of twisted Toeplitz matrices. Moreover, this proof does not need the first part above. Let

$$Z_N = \{-N, \dots, -1, 0, 1, \dots, N\}^d \subset \mathbb{Z}^d.$$

(Z_N) is a Følner sequence of $\mathbb{Z}^d$, that is,

$$\lim_{N\to\infty} \frac{|Z_N \triangle (Z_N + n)|}{|Z_N|} = 0, \quad \forall n \in \mathbb{Z}^d.$$

Define two maps A_N and B_N as follows:

$$A_N : \mathbb{T}^d_\theta \to \mathcal{B}(\ell_2^{|Z_N|}) \quad \text{with} \quad x \mapsto P_N([x]),$$

where $P_N : \mathcal{B}(\ell_2(\mathbb{Z}^d)) \to \mathcal{B}(\ell_2^{|Z_N|})$ with $(a_{mn}) \mapsto (a_{mn})_{m,n\in Z_N}$. And

$$B_N : \mathcal{B}(\ell_2^{|Z_N|}) \to \mathbb{T}^d_\theta \quad \text{with} \quad e_{mn} \mapsto \frac{1}{|Z_N|} e^{-\mathrm{i}n\tilde{\theta}(m-n)^t} U^{m-n}.$$

Here $\mathcal{B}(\ell_2^{|Z_N|})$ is endowed with the normalized trace. It is easy to check that both A_N, B_N are unital, completely positive and trace preserving. Consequently, A_N extends to a complete contraction from $L_p(\mathbb{T}^d_\theta)$ into $L_p(\mathcal{B}(\ell_2^{|Z_N|}))$, while B_N a complete contraction from $L_p(\mathcal{B}(\ell_2^{|Z_N|}))$ into $L_p(\mathbb{T}^d_\theta)$.

We now claim that $\lim_{N\to\infty} B_N \circ A_N(x) = x$ in $L_p(\mathbb{T}^d_\theta)$ for any $x \in L_p(\mathbb{T}^d_\theta)$. It suffices to consider a monomial $x = U^k$. Then

$$A_N(U^k) = (e^{\mathrm{i}n\tilde{\theta}(m-n)^t})_{m,n\in Z_N, m-n=k},$$

which implies

$$\begin{aligned} B_N \circ A_N(U^k) &= \frac{1}{|Z_N|} \sum_{m,n\in Z_N, m-n=k} U^{m-n} e^{-\mathrm{i}n\tilde{\theta}(m-n)^t} e^{in\tilde{\theta}(m-n)^t} \\ &= \frac{|Z_N \cap (Z_N + k)|}{|Z_N|} U^k. \end{aligned}$$

Then by the FØlner property of Z_N, we deduce that $\lim_N B_N \circ A_N(U^k) = U^k$ in $L_p(\mathbb{T}^d_\theta)$. So the claim is proved.

Now assume that the Schur multiplier $M_{\tilde{\phi}}$ is c.b. on $S_p(\ell_2(\mathbb{Z}^d))$. We want to prove that T_ϕ is c.b. on $L_p(\mathbb{T}^d_\theta)$. For any $x \in L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)$,

$$\|\mathrm{id} \otimes T_\phi(x)\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)} = \lim_N \|(\mathrm{id} \otimes B_N) \circ (\mathrm{id} \otimes A_N)(\mathrm{id} \otimes T_\phi(x))\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)}.$$

Using (3.6.2), we see that $\mathrm{id} \otimes A_N(\mathrm{id} \otimes T_\phi(x)) = \mathrm{id} \otimes M_{\tilde{\phi}}(\mathrm{id} \otimes A_N(x))$. Thus

$$\begin{aligned} \|\mathrm{id} \otimes T_\phi(x)\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)} &\le \limsup_N \|\mathrm{id} \otimes M_{\tilde{\phi}}(\mathrm{id} \otimes A_N(x))\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{B}(\ell_2^{|Z_N|}))} \\ &\le \limsup_N \|M_{\tilde{\phi}}\|_{\mathrm{cb}} \|\mathrm{id} \otimes A_N(x)\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{B}(\ell_2^{|Z_N|}))} \\ &\le \|M_{\tilde{\phi}}\|_{\mathrm{cb}} \|x\|_{L_p(\mathcal{B}(\ell_2)\overline{\otimes}\mathbb{T}^d_\theta)}. \end{aligned}$$

This implies that T_ϕ is c.b. on $L_p(\mathbb{T}^d_\theta)$ and $\|T_\phi\|_{\mathrm{cb}} \le \|M_{\tilde{\phi}}\|_{\mathrm{cb}}$, as desired.

In summary, we have proved that

$$T_\phi \text{ is c.b. on } L_p(\mathbb{T}^d_\theta) \iff M_{\tilde{\phi}} \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)).$$

Applying this result to the case of $\theta = 0$, we get that

$$T_\phi \text{ is c.b. on } L_p(\mathbb{T}^d) \iff M_{\tilde{\phi}} \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)).$$

Therefore,

$$\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta)) = \mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \quad \text{with equal norms.}$$

Thus the theorem is proved. □

Remark 3.6.4. The preceding proof shows that ϕ is a c.b. Fourier multiplier on $L_p(\mathbb{T}^d_\theta)$ iff $\tilde{\phi}$ is a c.b. Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$. This is the extension of Neuwirth and Ricard's transference result to twisted Toeplitz matrices. We will pursue this subject elsewhere for more general groups.

Remark 3.6.5. It would be interesting to study thin sets on $\mathbb{T}^d_\theta$, for instance, $\Lambda(p)$-sets and Sidon sets. At the level of complete boundedness, Theorem 3.6.3 shows that the $\Lambda(p)_{\mathrm{cb}}$-sets on $\mathbb{T}^d_\theta$ are exactly those on $\mathbb{T}^d$. We refer to Harcharras' thesis [29] for related results.

Theorem 3.6.3 suggests the following problem:

Problem 3.6.6. Let $2 < p \le \infty$. Does one have

$$\mathrm{M}(L_p(\mathbb{T}^d_\theta)) = \mathrm{M}(L_p(\mathbb{T}^d))\,?$$

We conjecture that the answer would be negative. Indeed, it is negative in the case of $p = \infty$ if one allows the number of generators to be infinite, as shown by the following remark that is communicated to us by Eric Ricard.

Remark 3.6.7. Let $\theta = (\theta_{kj})$ be the infinite skew matrix such that $\theta_{kj} = 1/2$ for all $k < j$. Let $\mathbb{T}_\theta^\infty$ be the associated quantum torus. Now the generators of $\mathbb{T}_\theta^\infty$ is a sequence $U = (U_1, U_2, \cdots)$ of anticommuting unitary operators:

$$U_k U_j = -U_j U_k, \quad \forall\, k \neq j.$$

Let ϕ be the indicator function of the subset $\Lambda = \{e_k \ : \ k \geq 1\}$ of $\mathbb{Z}^\infty$, where e_k is the element of $\mathbb{Z}^\infty$ whose coordinates all vanish except the one on the k-th position which is equal to 1. Then $\phi \in \mathrm{M}(L_\infty(\mathbb{T}_\theta^\infty))$ but $\phi \notin \mathrm{M}(L_\infty(\mathbb{T}^\infty))$.

Let us check this remark. Let $\alpha = (\alpha_k) \subset \mathbb{C}$ be a finite sequence and set

$$x = \sum_k \alpha_k U_k.$$

Then by the anticommuting relation we have

$$x^*x + xx^* = 2\sum_k |\alpha_k|^2 + \sum_{j\neq k} \bar{\alpha}_j \alpha_k (U_j^* U_k + U_k U_j^*) = 2\sum_k |\alpha_k|^2.$$

It then follows that

$$\|x\|_\infty \leq \sqrt{2}\, \|\alpha\|_2.$$

On the other hand, it is clear that

$$\|x\|_\infty \geq \|x\|_2 \geq \|\alpha\|_2.$$

We then deduce that for any $\alpha = (\alpha_k) \subset \mathbb{C}$ the series $\sum_k \alpha_k U_k$ converges in $\mathbb{T}_\theta^\infty$ iff $\alpha \in \ell_2$. In this case, we have

$$\|\alpha\|_2 \leq \big\| \sum_k \alpha_k U_k \big\|_\infty \leq \sqrt{2}\, \|\alpha\|_2.$$

This clearly implies that ϕ is a bounded L_∞ multiplier on $\mathbb{T}_\theta^\infty$. However, ϕ is not a bounded L_∞ multiplier on $\mathbb{T}^\infty$. Otherwise, the closed subspace of $L_\infty(\mathbb{T}^\infty)$ generated by the generators $(z_1, z_2, \cdots)$ would be complemented in $L_\infty(\mathbb{T}^\infty)$. But this subspace is isometric to ℓ_1. It is well known that ℓ_1 cannot be isomorphic to a complemented subspace of an L_∞-space. This contradiction yields that $\phi \notin \mathrm{M}(L_\infty(\mathbb{T}^\infty))$. This example also shows that

$$\mathrm{M}_{\mathrm{cb}}(L_\infty(\mathbb{T}_\theta^\infty)) \subsetneqq \mathrm{M}(L_\infty(\mathbb{T}_\theta^\infty)),$$

in contrast with equality (3.6.3) in the commutative case.

We end this section by showing the equality $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = \mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d))$ in Theorem 3.6.3 holds completely isometrically. To this end we first need to equip these spaces with an operator space structure. Recall that for two operator spaces E and F the space $\mathrm{CB}(E, F)$ has a natural operator space structure by setting $\mathbb{M}_n(\mathrm{CB}(E, F)) = \mathrm{CB}(E, \mathbb{M}_n(F))$. Then $\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$ inherits the operator space structure of $\mathrm{CB}(L_p(\mathbb{T}_\theta^d),\, L_p(\mathbb{T}_\theta^d))$. Let $\mathrm{TM}_{\mathrm{cb}}(S_p(\ell_2(\mathbb{Z}^d)))$ be the subspace of all c.b. Schur multipliers ψ on $S_p(\ell_2(\mathbb{Z}^d))$ which are of the Toeplitz form, i.e., $\psi_{mn} = \phi_{m-n}$ for some ϕ. $\mathrm{TM}_{\mathrm{cb}}(S_p(\ell_2(\mathbb{Z}^d)))$ is also an operator space via $\mathrm{TM}_{\mathrm{cb}}(S_p(\ell_2(\mathbb{Z}^d))) \subset \mathrm{CB}(S_p(\ell_2(\mathbb{Z}^d)), S_p(\ell_2(\mathbb{Z}^d)))$.

Proposition 3.6.8. *Let $1 \leq p \leq \infty$. Then*

$$\mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = \mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \cong \mathrm{TM}_{\mathrm{cb}}(S_p(\ell_2(\mathbb{Z}^d)))$$

completely isometrically, where the last identification is realized by $\phi \in \mathrm{M}_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \leftrightarrow \tilde{\phi} \in \mathrm{TM}_{\mathrm{cb}}(S_p(\ell_2(\mathbb{Z}^d)))$ with $\tilde{\phi}_{mn} = \phi_{m-n}$.

Proof. We require the following elementary fact: Let $\mathcal{M}$ be a von Neumann algebra and u a unitary operator in $\mathbb{M}_n(\mathcal{M})$. Then for any $x \in \mathbb{M}_n(L_p(\mathcal{M}))$

$$\|uxu^*\|_{\mathbb{M}_n(L_p(\mathcal{M}))} = \|x\|_{\mathbb{M}_n(L_p(\mathcal{M}))}.$$

Indeed, this is obvious for $p = \infty$. Then by duality, it is also true for $p = 1$. Finally, by interpolation, we deduce this equality for any $1 < p < \infty$. Armed with this fact, we can modify the proof of Theorem 3.6.3 to get the announced assertion. The details are left to the reader. □

3.7 Hardy spaces

There exist several ways to define Hardy spaces on quantum tori. The resulting spaces may be different. The approach that we adopt in this section is based on the Littlewood-Paley theory and real variable method in Fourier analysis. Our Hardy spaces are defined by square functions in terms of the circular Poisson semigroup $\mathbb{P}_r$. This allows us to use the recent developments of operator-valued harmonic analysis and noncommutative Littlewood-Paley-Stein theory.

For any $x \in \mathbb{T}_\theta^d$ define

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr \Big)^{1/2}.$$

For $1 \le p < \infty$ let

$$\|x\|_{\mathrm{H}_p^c} = |\hat{x}(\mathbf{0})| + \|G_c(x)\|_{L_p(\mathbb{T}_\theta^d)}.$$

This is a norm on $\mathbb{T}_\theta^d$ (cf. e.g. [37]). We define the column Hardy space $\mathrm{H}_p^c(\mathbb{T}_\theta^d)$ as the completion of $\mathbb{T}_\theta^d$ with respect to this norm. The row Hardy space $\mathrm{H}_p^r(\mathbb{T}_\theta^d)$ is defined to be the space of all x such that $x^* \in \mathrm{H}_p^c(\mathbb{T}_\theta^d)$ equipped with the natural norm. The mixture Hardy spaces are defined as follows: If $1 \le p < 2$,

$$\mathrm{H}_p(\mathbb{T}_\theta^d) = \mathrm{H}_p^c(\mathbb{T}_\theta^d) + \mathrm{H}_p^r(\mathbb{T}_\theta^d)$$

equipped with the sum norm

$$\|x\|_{\mathrm{H}_p} = \inf\{\|a\|_{\mathrm{H}_p^c} + \|b\|_{\mathrm{H}_p^r} \;:\; x = a + b, a \in \mathrm{H}_p^c(\mathbb{T}_\theta^d), b \in \mathrm{H}_p^r(\mathbb{T}_\theta^d)\},$$

and if $2 \le p < \infty$,

$$\mathrm{H}_p(\mathbb{T}_\theta^d) = \mathrm{H}_p^c(\mathbb{T}_\theta^d) \cap \mathrm{H}_p^r(\mathbb{T}_\theta^d)$$

equipped with the intersection norm

$$\|x\|_{\mathrm{H}_p} = \max\{\|x\|_{\mathrm{H}_p^c},\ \|x\|_{\mathrm{H}_p^r}\}.$$

We will also study the BMO spaces over $\mathbb{T}_\theta^d$. Set

$$\mathrm{BMO}^c(\mathbb{T}_\theta^d) = \{x \in L_2(\mathbb{T}_\theta^d) \;:\; \sup_r \big\|\mathbb{P}_r\big[|x - \mathbb{P}_r[x]|^2\big]\big\|_\infty < \infty\}$$

equipped with the norm

$$\|x\|_{\mathrm{BMO}^c} = \max\{|\hat{x}(\mathbf{0})|,\ \sup_r \big\|\mathbb{P}_r\big[|x - \mathbb{P}_r[x]|^2\big]\big\|_\infty^{1/2}\}.$$

$\mathrm{BMO}^r(\mathbb{T}^d_\theta)$ is defined as the space of all x such that $x^* \in \mathrm{BMO}^c(\mathbb{T}^d_\theta)$ with the norm $\|x\|_{\mathrm{BMO}^r} = \|x^*\|_{\mathrm{BMO}^c}$. The mixture $\mathrm{BMO}(\mathbb{T}^d_\theta)$ is the intersection of these two spaces:

$$\mathrm{BMO}(\mathbb{T}^d_\theta) = \mathrm{BMO}^c(\mathbb{T}^d_\theta) \cap \mathrm{BMO}^r(\mathbb{T}^d_\theta)$$

with the intersection norm.

The above definitions are motivated by Hardy spaces of noncommutative martingales ([43, 69]) and of quantum Markov semigroups ([37, 38, 52]). The main results of this section are summarized in the following statement which shows that the Hardy spaces on $\mathbb{T}^d_\theta$ possess the properties of the usual Hardy spaces, as expected.

Theorem 3.7.1. i) *Let $1 < p < \infty$. Then $\mathrm{H}_p(\mathbb{T}^d_\theta) = L_p(\mathbb{T}^d_\theta)$ with equivalent norms.*

ii) *The dual space of $\mathrm{H}^c_1(\mathbb{T}^d_\theta)$ is equal to $\mathrm{BMO}^c(\mathbb{T}^d_\theta)$ with equivalent norms via the duality bracket*

$$\langle x,\ y\rangle = \tau(xy^*), \quad x \in L_2(\mathbb{T}^d_\theta),\ y \in \mathrm{BMO}^c(\mathbb{T}^d_\theta).$$

The same assertion holds for the row and mixture spaces too.

iii) *Let $1 < p < \infty$. Then*

$$(\mathrm{BMO}^c(\mathbb{T}^d_\theta),\ \mathrm{H}^c_1(\mathbb{T}^d_\theta))_{1/p} = \mathrm{H}^c_p(\mathbb{T}^d_\theta) = (\mathrm{BMO}^c(\mathbb{T}^d_\theta),\ \mathrm{H}^c_1(\mathbb{T}^d_\theta))_{1/p,p}$$

with equivalent norms, where $(\,\cdot\,,\,\cdot\,)_{1/p}$ and $(\,\cdot\,,\,\cdot\,)_{1/p,p}$ denote respectively the complex and real interpolation functors.

iv) *Let $1 < p < \infty$ and $X_0 \in \{\mathrm{BMO}(\mathbb{T}^d_\theta),\ L_\infty(\mathbb{T}^d_\theta)\}$, $X_1 \in \{\mathrm{H}_1(\mathbb{T}^d_\theta),\ L_1(\mathbb{T}^d_\theta)\}$. Then*

$$(X_0,\ X_1)_{1/p} = L_p(\mathbb{T}^d_\theta) = (X_0,\ X_1)_{1/p,p}$$

with equivalent norms.

Some parts of this theorem can be deduced from existing results in literature. This is the case of i) and the complex interpolation equality $(\mathrm{BMO}(\mathbb{T}^d_\theta),\ L_1(\mathbb{T}^d_\theta))_{1/p} = L_p(\mathbb{T}^d_\theta)$ in iv). Let us explain these two points.

According to the discussion following Theorem 3.3.2, the circular Poisson semigroup $(\mathbb{P}_r)_{0\le r<1}$ on $\mathbb{T}^d_\theta$ is a noncommutative symmetric diffusion semigroup in the sense of [45]. We claim that $(\mathbb{P}_r)_{0\le r<1}$ admits a Markov dilation (as well as a Rota dilation) in the sense of [37]. Indeed, considering the von Neumann subalgebra $\widetilde{\mathbb{T}^d_\theta}$ of $\mathcal{N}_\theta = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathbb{T}^d_\theta$, which is the image of $\mathbb{T}^d_\theta$ under the map $x \mapsto \tilde{x}$, we see that the circular Poisson semigroup on the usual torus $\mathbb{T}^d$ extends to a semigroup by tensoring with $\mathrm{id}_{\mathbb{T}^d_\theta}$. By a slight abuse of notation, we will also use $(\mathbb{P}_r)_{0\le r<1}$ to denote the circular Poisson semigroup on the usual torus $\mathbb{T}^d$. It is clear that $\mathbb{P}_r \otimes \mathrm{id}_{\mathbb{T}^d_\theta}[\tilde{x}] = \widetilde{\mathbb{P}_r[x]}$ for any $x \in \mathbb{T}^d_\theta$. Since every symmetric diffusion semigroup on a commutative von Neumann algebra can be dilated to a Markov unitary group as well as an inverse martingale, $(\mathbb{P}_r \otimes \mathrm{id}_{\mathbb{T}^d_\theta})_{0\le r<1}$ admits a Markov/Rota dilation, so does its restriction to $\widetilde{\mathbb{T}^d_\theta}$. Our claim then follows. Therefore, the semigroup $(\mathbb{P}_r)_{0\le r<1}$ on $\mathbb{T}^d_\theta$ satisfies the assumption of [37] which insures the existence of an associated H_∞-functional calculus. Thus by [37, Theorem 7.6], we get i). On the other hand, the interpolation theorem of [38] yields $(\mathrm{BMO}(\mathbb{T}^d_\theta),\ L_1(\mathbb{T}^d_\theta))_{1/p} = L_p(\mathbb{T}^d_\theta)$. We also point out that the duality result in part ii) could be deduced from a work in progress of Avsec and Mei [3].

To prove the remaining parts of Theorem 3.7.1 we will use transference to reduce the problem to the corresponding one on $\mathcal{N}_\theta$ and then use Mei's results [52]. An advantage of this proof is that it also provides an alternative (more elementary) approach to the two parts already considered in the previous paragraph. Recall that the framework of [52] is the Euclidean space $\mathbb{R}^d$, and the Hardy spaces there are defined by using the Poisson semigroup on $\mathbb{R}^d$. The geometry of $\mathbb{R}^d$ is simpler than $\mathbb{T}^d$. But what really renders matters more handy in $\mathbb{R}^d$ is the explicit compact formula of the Poisson kernel (or its growth estimates). The situation for $\mathbb{T}^d$ is harder. Although it is claimed in [52] as remarks that all results there hold equally with essentially the same proofs in the d-torus setting, this claim is clearly true for $\mathbb{T}$ thanks to the explicit simple formula of the Poisson kernel of $\mathbb{T}$. Howeve r, it would not be so transparent whenever $d \geq 2$. As a byproduct of our proof below of Theorem 3.7.1, we remedy this situation, which constitutes another advantage of our approach via transference. Finally, it seems that even in the scalar case there does not exist published references on Hardy space theory on $\mathbb{T}^d$ for $d \geq 2$ via the Littlewood-Paley theory, although this theory is certainly known as folklore to many specialists. Our approach provides, in particular, a complete picture of the scalar-valued Hardy space theory on $\mathbb{T}^d$, exactly parallel to that on $\mathbb{R}^d$.

Convention. For notational simplicity we will denote all circular Poisson semigroups considered in the sequel by $(\mathbb{P}_r)_{0\leq r<1}$. Thus $\mathbb{P}_r \otimes \mathrm{id}_{\mathbb{T}^d_\theta}$ will be simply denoted by $\mathbb{P}_r$. This slight abuse of notation should not cause any confusion in concrete contexts. For instance, for $x \in \mathbb{T}^d_\theta$, $\mathbb{P}_r[x] \in \mathbb{T}^d_\theta$ while for $f \in \mathcal{N}_\theta$, $\mathbb{P}_r[f] = \mathbb{P}_r \otimes \mathrm{id}_{\mathbb{T}^d_\theta}[f] \in \mathcal{N}_\theta$. On the other hand, $\mathbb{P}_r$ will also stands for the circular Poisson kernel on $\mathbb{T}^d$ given by (3.2.3). Thus for $f \in L_1(\mathcal{N}_\theta)$ we have (recalling that dm denotes Haar measure on $\mathbb{T}^d$)

$$\mathbb{P}_r[f](z) = \mathbb{P}_r * f(z) = \int_{\mathbb{T}^d} \mathbb{P}_r(z \cdot \overline{w}) f(w) dm(w), \quad z \in \mathbb{T}^d.$$

We will study several BMO norms as well as H_p^c norms. The notational system for these norms (or spaces) might look heavy; but everything should be clear in concrete contexts. We start our analysis with BMO spaces on $\mathbb{T}^d$ with values in a von Neumann algebra $\mathcal{M}$. For simplicity we will assume that $\mathcal{M}$ is equipped with a normal faithful tracial state τ ($\mathcal{M}$ will be $\mathbb{T}^d_\theta$ in the proof of Theorem 3.7.1) . We start with the BMO space. Let

$$\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M}) = \{f \in L_2(\mathbb{T}^d; L_2(\mathcal{M})) \ : \ \sup_r \big\|\mathbb{P}_r\big[|f - \mathbb{P}_r[f]|^2\big]\big\|_\infty < \infty\},$$

equipped with the norm

$$\|f\|_{\mathrm{BMO}^c} = \max\big\{\|\hat{f}(\mathbf{0})\|_\infty,\ \sup_r \big\|\mathbb{P}_r\big[|f - \mathbb{P}_r[f]|^2\big]\big\|_\infty^{1/2}\big\}.$$

Here the first L_∞-norm is the one of $\mathcal{M}$ and the second that of $L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$.

We require the following lemma which characterizes $\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})$ by the noncommutative analogue of the usual Garsia norm. This lemma is a special case of [38, Theorem 2.9]. But we prefer to present the following elementary proof which was communicated to us by Tao Mei.

Lemma 3.7.2. *For any $f \in L_2(\mathbb{T}^d; L_2(\mathcal{M}))$ we have*

$$\sup_r \big\|\mathbb{P}_r\big[|f - \mathbb{P}_r[f]|^2\big]\big\|_\infty \approx \sup_r \big\|\mathbb{P}_r[|f|^2] - |\mathbb{P}_r[f]|^2\big\|_\infty \tag{3.7.1}$$

with universal equivalence constants.

Proof. First note that

$$\mathbb{P}_r[|f|^2](z) - |\mathbb{P}_r[f]|^2(z) = \mathbb{P}_r[|f - \mathbb{P}_r[f](z)|^2](z), \quad \forall\, z \in \mathbb{T}^d.$$

Thus

$$\sup_{0\le r<1} \big\|\mathbb{P}_r[|f|^2] - |\mathbb{P}_r[f]|^2\big\|_\infty = \sup_{0\le r<1} \sup_{z\in\mathbb{T}^d} \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f](z)|^2](z)\big\|_{\mathcal{M}}. \tag{3.7.2}$$

The right hand side is exactly the analogue of the usual Garsia norm (cf. [25, Corollary VI.2.4]). For any fixed r and z we have

$$\big\|\mathbb{P}_r[|f - \mathbb{P}_r[f]|^2](z)\big\|_{\mathcal{M}}^{1/2} \le \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f](z)|^2](z)\big\|_{\mathcal{M}}^{1/2} + \big\|\mathbb{P}_r[\big|\mathbb{P}_r[f - \mathbb{P}_r[f](z)]\big|^2](z)\big\|_{\mathcal{M}}^{1/2}.$$

By Kadison's Cauchy-Schwarz inequality,

$$\mathbb{P}_r\big[\big|\mathbb{P}_r[f - \mathbb{P}_r[f](z)]\big|^2\big] \le \mathbb{P}_{r^2}\big[\big|f - \mathbb{P}_r[f](z)\big|^2\big].$$

On the other hand, since $\mathbb{P}_r$ is subordinated to the heat semigroup on $\mathbb{T}^d$, by the subordination formula, one has $\mathbb{P}_{r^2}[g] \le 2\mathbb{P}_r[g]$ for positive $g \in L_1(\mathbb{T}^d; L_1(\mathcal{M}))$. Alternatively, this inequality can be easily checked by (3.7.7) below. Then we deduce that

$$\sup_r \sup_z \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f]|^2](z)\big\|_{\mathcal{M}}^{1/2} \le (1+\sqrt{2}\,) \sup_r \sup_z \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f](z)|^2](z)\big\|_{\mathcal{M}}^{1/2}.$$

This is the upper estimate of (3.7.1).

The converse inequality is harder. Fix $f \in L_2(\mathbb{T}^d; L_2(\mathcal{M}))$. By triangle inequality, we have

$$\begin{aligned}\big\|\mathbb{P}_r[|f|^2] - |\mathbb{P}_r[f]|^2\big\|_\infty^{1/2} \le& \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f]|^2] - |\mathbb{P}_r[f - \mathbb{P}_r[f]]|^2\big\|_\infty^{1/2}\\ &+ \big\|\mathbb{P}_r[|\mathbb{P}_r[f]|^2] - |\mathbb{P}_r[\mathbb{P}_r[f]]|^2\big\|_\infty^{1/2}\\ \le& \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f]|^2]\big\|_\infty^{1/2}\\ &+ \big\|\mathbb{P}_r[|\mathbb{P}_r[f]|^2] - |\mathbb{P}_r[\mathbb{P}_r[f]]|^2\big\|_\infty^{1/2}.\end{aligned}$$

Assuming for the moment the following inequality

$$2\mathbb{P}_r[|\mathbb{P}_r[f]|^2] \le \mathbb{P}_{r^2}[|f|^2] + |\mathbb{P}_{r^2}[f]|^2, \tag{3.7.3}$$

we get

$$2\big(\mathbb{P}_r[|\mathbb{P}_r[f]|^2] - |\mathbb{P}_{r^2}[f]|^2\big) \le \mathbb{P}_{r^2}[|f|^2] - |\mathbb{P}_{r^2}[f]|^2.$$

Combining the preceding inequalities, we then deduce that

$$\sup_r \big\|\mathbb{P}_r[|f|^2] - |\mathbb{P}_r[f]|^2\big\|_\infty^{1/2} \le \sup_r \big\|\mathbb{P}_r[|f - \mathbb{P}_r[f]|^2]\big\|_\infty^{1/2} + \frac{1}{\sqrt{2}} \sup_r \big\|\mathbb{P}_r[|f|^2] - |\mathbb{P}_r[f]|^2\big\|_\infty^{1/2}.$$

Whence the lower estimate of (3.7.1) with $2+\sqrt{2}$ as constant.

It remains to prove (3.7.3). To this end, it is more convenient to work with $\mathbb{Q}_\varepsilon = \mathbb{P}_r$ for $r = e^{-2\pi\varepsilon}$. Then we must show

$$\mathbb{Q}_\varepsilon[|\mathbb{Q}_\varepsilon[f]|^2] - |\mathbb{Q}_{2\varepsilon}[f]|^2 \le \mathbb{Q}_{2\varepsilon}[|f|^2] - \mathbb{Q}_\varepsilon[|\mathbb{Q}_\varepsilon[f]|^2], \quad \forall\, \varepsilon > 0. \tag{3.7.4}$$

Let us write

$$\mathbb{Q}_\varepsilon[|\mathbb{Q}_\varepsilon[f]|^2] - |\mathbb{Q}_{2\varepsilon}[f]|^2 = -\int_0^\varepsilon \frac{d}{dt}\mathbb{Q}_{\varepsilon-t}[|\mathbb{Q}_{\varepsilon+t}[f]|^2]dt.$$

Let A be the negative generator of $\mathbb{Q}_\varepsilon$: $\mathbb{Q}_\varepsilon = e^{-\varepsilon A}$. Then

$$\begin{aligned}\frac{d}{dt}\mathbb{Q}_{\varepsilon-t}[|\mathbb{Q}_{\varepsilon+t}[f]|^2] =& A\mathbb{Q}_{\varepsilon-t}[|\mathbb{Q}_{\varepsilon+t}[f]|^2]\\ &- \mathbb{Q}_{\varepsilon-t}\big[(A\mathbb{Q}_{\varepsilon+t}[f]^*)(\mathbb{Q}_{\varepsilon+t}[f]) + (\mathbb{Q}_{\varepsilon+t}[f]^*)(A\mathbb{Q}_{\varepsilon+t}[f])\big].\end{aligned}$$

For $s > 0$ let

$$F_s(g) = -A\mathbb{Q}_s[|\mathbb{Q}_s[g]|^2] + \mathbb{Q}_s\big[(A\mathbb{Q}_s[g]^*)(\mathbb{Q}_s[g]) + (\mathbb{Q}_s[g]^*)(A\mathbb{Q}_s[g])\big].$$

Then for $g = \mathbb{Q}_{\varepsilon+t}[f]$ we have

$$\mathbb{Q}_\varepsilon[|\mathbb{Q}_\varepsilon[f]|^2] - |\mathbb{Q}_{2\varepsilon}[f]|^2 = \lim_{s\to 0}\int_0^\varepsilon \mathbb{Q}_{\varepsilon-t}[F_s(g)]dt. \tag{3.7.5}$$

It is easy to check that $\lim_{s\to\infty} F_s(g) = 0$ (one can use, for instance, (3.7.7) below). Then

$$F_s(g) = -\int_s^\infty \frac{d}{du}F_u(g)du. \tag{3.7.6}$$

Elementary calculations lead to

$$\begin{aligned}\frac{d}{du}F_u(g) &= A^2\mathbb{Q}_u[|\mathbb{Q}_u[g]|^2] - \mathbb{Q}_u\big[(A^2\mathbb{Q}_u[g]^*)(\mathbb{Q}_u[g]) + (\mathbb{Q}_u[g]^*)(A^2\mathbb{Q}_u[g])\big] - 2\mathbb{Q}_u[|A\mathbb{Q}_u[g]|^2]\\ &= \mathbb{Q}_u\big[A^2|\mathbb{Q}_u[g]|^2 - (A^2\mathbb{Q}_u[g]^*)(\mathbb{Q}_u[g]) - (\mathbb{Q}_u[g]^*)(A^2\mathbb{Q}_u[g]) - 2|A\mathbb{Q}_u[g]|^2\big].\end{aligned}$$

Note that

$$A = 2\pi\sqrt{-\mathbb{D}}\,,$$

where $\mathbb{D}$ is the Laplacian of $\mathbb{T}^d$:

$$\mathbb{D} = \sum_{k=1}^d \frac{\partial^2}{\partial z_k^2}\,.$$

So $A^2 = -4\pi^2\mathbb{D}$ and

$$A^2|\mathbb{Q}_u[g]|^2 = (A^2\mathbb{Q}_u[g]^*)(\mathbb{Q}_u[g]) + (\mathbb{Q}_u[g]^*)(A^2\mathbb{Q}_u[g]) - 8\pi^2\sum_{k=1}^d\Big|\frac{\partial}{\partial z_k}\mathbb{Q}_u[g]\Big|^2.$$

Therefore,

$$\frac{d}{du}F_u(g) = -8\pi^2\sum_{k=1}^d \mathbb{Q}_u\big[\big|\frac{\partial}{\partial z_k}\mathbb{Q}_u[g]\big|^2\big] - 2\mathbb{Q}_u\big[|A\mathbb{Q}_u[g]|^2\big].$$

Recall that $g = \mathbb{Q}_{\varepsilon+t}[f]$. By Kadison's Cauchy-Schwarz inequality and using the above equality twice, we obtain

$$-\frac{d}{du}F_u(g) \le \mathbb{Q}_\varepsilon\Big[8\pi^2\sum_{k=1}^d\mathbb{Q}_u\Big[\Big|\frac{\partial}{\partial z_k}\mathbb{Q}_u[h]\Big|^2\Big] + 2\mathbb{Q}_u\big[|A\mathbb{Q}_u[h]|^2\big]\Big] \le -\mathbb{Q}_\varepsilon\Big[\frac{d}{du}F_u(h)\Big],$$

where $h = \mathbb{Q}_t[f]$. Thus by (3.7.6),

$$F_s(g) \le \mathbb{Q}_\varepsilon[F_s(h)].$$

Hence by (3.7.5) and inverting the procedure leading to (3.7.5), we obtain

$$\begin{aligned}\mathbb{Q}_\varepsilon[|\mathbb{Q}_\varepsilon[f]|^2] - |\mathbb{Q}_{2\varepsilon}[f]|^2 &\le \lim_{s\to 0}\int_0^\varepsilon \mathbb{Q}_{2\varepsilon-t}[F_s(h)]dt\\ &= -\int_0^\varepsilon \frac{\partial}{\partial t}\mathbb{Q}_{2\varepsilon-t}[|\mathbb{Q}_t[f]|^2]dt = \mathbb{Q}_{2\varepsilon}[|f|^2] - \mathbb{Q}_\varepsilon[\mathbb{Q}_\varepsilon[f]|^2].\end{aligned}$$

This yields (3.7.4), and (3.7.3) too. Thus the lemma is proved. $\square$

Although this is not really necessary, it is more convenient to work with the cube $\mathbb{I}^d = [0, 1]^d$ instead of $\mathbb{T}^d$. Another reason is that the case of $\mathbb{I}^d$ is closer to that of $\mathbb{R}^d$. So we will identify $\mathbb{T}^d$ with $\mathbb{I}^d$, as in the proof of Theorem 3.3.2. The addition in $\mathbb{I}^d$ is modulo 1 coordinatewise, which corresponds to the multiplication in $\mathbb{T}^d$ under the identification $(e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}) \leftrightarrow (s_1, \cdots, s_d)$. Accordingly, functions on $\mathbb{T}^d$ and $\mathbb{I}^d$ are identified too. Thus $L_p(\mathbb{T}^d; L_p(\mathcal{M})) = L_p(\mathbb{I}^d; L_p(\mathcal{M}))$.

We will use the following Poisson summation formula (see [83, Corollary VII.2.6]):

$$\mathbb{P}_r(z) = \sum_{m\in\mathbb{Z}^d} \varphi_\varepsilon(s+m) \quad \text{with} \quad z = (e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}) \quad \text{and} \quad r = e^{-2\pi\varepsilon}, \tag{3.7.7}$$

where φ_ε is the Poisson kernel on $\mathbb{R}^d$:

$$\varphi_\varepsilon(s) = c_d \, \frac{\varepsilon}{(\varepsilon^2 + |s|^2)^{(d+1)/2}}\,, \quad s = (s_1, \cdots, s_d) \in \mathbb{R}^d.$$

In the sequel, we will always assume that z and s, r and ε are related as in (3.7.7). Let

$$\mathbb{Q}_\varepsilon(s) = \sum_{m\in\mathbb{Z}^d} \varphi_\varepsilon(s+m), \quad s \in \mathbb{I}^d. \tag{3.7.8}$$

This notation is consistent with that introduced during the proof of Lemma 3.7.2 since

$$\mathbb{P}_r[f](z) = \mathbb{Q}_\varepsilon[f](s) = \mathbb{Q}_\varepsilon * f(s) = \int_{\mathbb{I}^d} \mathbb{Q}_\varepsilon(s-t) f(t)dt. \tag{3.7.9}$$

An interval of $\mathbb{I}$ is either a subinterval of $\mathbb{I}$ or a union $[b, 1] \cup [0, a]$ with $0 < a < b < 1$. The latter union is the interval $[b-1, a]$ by the addition modulo 1 of $\mathbb{I}$. So the intervals of $\mathbb{I}$ correspond exactly to the arcs of $\mathbb{T}$. A cube of $\mathbb{I}^d$ is a product of d intervals. For $f \in L_1(\mathbb{I}^d; L_1(\mathcal{M}))$ and a cube $Q \subset \mathbb{I}^d$ let

$$f_Q = \frac{1}{|Q|} \int_Q f ds,$$

where $|Q|$ denotes the volume of Q. Then we define $\mathrm{BMO}^c(\mathbb{I}^d; \mathcal{M})$ as the space of all $f \in L_2(\mathbb{I}^d; L_2(\mathcal{M}))$ such that

$$\sup_{Q\subset\mathbb{I}^d\text{cube}} \Big\| \frac{1}{|Q|} \int_Q |f - f_Q|^2 ds \Big\|_\infty < \infty,$$

equipped with the norm

$$\|f\|_{\mathrm{BMO}^c} = \max\big\{ \|f_{\mathbb{I}^d}\|_\infty, \quad \sup_{Q\subset\mathbb{I}^d\text{cube}} \Big\| \frac{1}{|Q|} \int_Q |f - f_Q|^2 ds \Big\|_\infty^{1/2} \big\}.$$

Here $\|\ \|_\infty$ denotes, of course, the norm of $\mathcal{M}$.

Lemma 3.7.3. $\mathrm{BMO}^c(\mathbb{T}^d; \mathcal{M}) = \mathrm{BMO}^c(\mathbb{I}^d; \mathcal{M})$ *with equivalent norms.*

Proof. Fix $f \in L_2(\mathbb{T}^d; L_2(\mathcal{M}))$. Without loss of generality, assume that $\hat{f}(\mathbf{0}) = f_{\mathbb{I}^d} = 0$. By Lemma 3.7.2 and (3.7.9), we need to show

$$\sup_{\varepsilon>0} \sup_{s\in\mathbb{I}^d} \big\| \mathbb{Q}_\varepsilon \big[|f - \mathbb{Q}_\varepsilon[f](s)|^2 \big](s) \big\|_\infty \approx \sup_{Q\subset\mathbb{I}^d\text{cube}} \Big\| \frac{1}{|Q|} \int_Q |f - f_Q|^2 dt \Big\|_\infty. \tag{3.7.10}$$

Let Q be a cube of $\mathbb{I}^d$. Let s and ε be the center and half of the side length of Q, respectively. It is clear that

$$\frac{1}{|Q|}\,\mathbb{1}_Q(t) \leq C_d\,\varphi_\varepsilon(s-t) \leq C_d\,\mathbb{Q}_\varepsilon(s-t).$$

Thus

$$\frac{1}{|Q|}\int_Q \big|f(t)-\mathbb{Q}_\varepsilon[f](s)\big|^2 dt \leq C_d\,\mathbb{Q}_\varepsilon\big[|f-\mathbb{Q}_\varepsilon[f](s)|^2\big](s).$$

Then

$$\frac{1}{|Q|}\int_Q \big|f-f_Q\big|^2 dt \leq 4\frac{1}{|Q|}\int_Q \big|f-\mathbb{Q}_\varepsilon[f](s)\big|^2 dt \leq 4C_d\,\mathbb{Q}_\varepsilon\big[|f-\mathbb{Q}_\varepsilon[f](s)|^2\big](s).$$

This yields one inequality of (3.7.10).

To show the converse inequality fix $s\in\mathbb{I}^d$ and $\varepsilon>0$. Consider first the case $\varepsilon\geq 1/2$. Then $\mathbb{Q}_\varepsilon(t)\approx 1$ for any $t\in\mathbb{I}^d$. It follows that

$$\mathbb{Q}_\varepsilon\big[|f-\mathbb{Q}_\varepsilon[f](s)|^2\big](s) \approx \int_{\mathbb{I}^d}|f-\mathbb{Q}_\varepsilon[f](s)|^2 \lesssim \int_{\mathbb{I}^d}|f|^2.$$

Whence

$$\big\|\mathbb{Q}_\varepsilon\big[|f-\mathbb{Q}_\varepsilon[f](s)|^2\big](s)\big\|_\infty \lesssim \Big\|\int_{\mathbb{I}^d}|f|^2\Big\|_\infty \leq \|f\|^2_{\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})}.$$

Now assume $\varepsilon<1/2$. By the proof of Theorem 3.3.2, for any $t\in\mathbb{I}^d$

$$\sum_{m\neq\mathbf{0}}\varphi_\varepsilon(t+m)\lesssim\varepsilon\lesssim\varphi_\varepsilon(t)\,.$$

Consequently,

$$\mathbb{Q}_\varepsilon\big[|f-\mathbb{Q}_\varepsilon[f](s)|^2\big](s) \lesssim \int_{\mathbb{I}^d}\varphi_\varepsilon(s-t)|f(t)-\mathbb{Q}_\varepsilon[f](s)|^2 dt.$$

Let $Q=\{t\in\mathbb{I}^d\ :\ |t-s|\leq\varepsilon\}$ and $Q_k=\{t\in\mathbb{I}^d\ :\ |t-s|\leq 2^{k+1}\varepsilon\}$. Then

$$\begin{aligned}\int_{\mathbb{I}^d}\varphi_\varepsilon(s-t)|f(t)-f_Q|^2dt &= \int_Q\varphi_\varepsilon(s-t)|f(t)-f_Q|^2dt\\ &\quad+\sum_{k\geq0}\int_{2^k\varepsilon<|t-s|\leq 2^{k+1}\varepsilon}\varphi_\varepsilon(s-t)|f(t)-f_Q|^2dt\\ &\lesssim \frac{1}{|Q|}\int_Q|f(t)-f_Q|^2+\sum_{k\geq0}\frac{1}{2^k|Q_k|}\int_{Q_k}|f(t)-f_Q|^2dt.\end{aligned}$$

The above sums on k are in fact finite sums. By triangle inequality (with $Q_{-1}=Q$),

$$\Big\|\frac{1}{|Q_k|}\int_{Q_k}|f-f_Q|^2\Big\|_\infty^{1/2} \leq \Big\|\frac{1}{|Q_k|}\int_{Q_k}|f-f_{Q_k}|^2\Big\|_\infty^{1/2}+\sum_{j=0}^{k}\|f_{Q_j}-f_{Q_{j-1}}\|_\infty.$$

However,

$$\begin{aligned}\|f_{Q_j}-f_{Q_{j-1}}\|_\infty^2 &\leq \Big\|\frac{1}{|Q_{j-1}|}\int_{Q_{j-1}}|f-f_{Q_j}|^2\Big\|_\infty\\ &\leq 2^d\Big\|\frac{1}{|Q_j|}\int_{Q_j}|f-f_{Q_j}|^2\Big\|_\infty \leq 2^d\|f\|^2_{\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})}.\end{aligned}$$

Combining the preceding inequalities, we obtain

$$\|\mathbb{Q}_\varepsilon[|f-f_Q|^2](s)\|_\infty \lesssim \sum_{k\geq 0}\frac{k+1}{2^k}\,\|f\|^2_{\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})} \lesssim \|f\|^2_{\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})}.$$

Finally,

$$\|\mathbb{Q}_\varepsilon[|f-\mathbb{Q}_\varepsilon[f](s)|^2](s)\|_\infty^{1/2} \leq 2\|\mathbb{Q}_\varepsilon[|f-f_Q|^2](s)\|_\infty^{1/2} \lesssim \|f\|_{\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})}.$$

This implies the missing inequality of (3.7.10). □

Remark 3.7.4. The previous proof shows implicitly that the supremum on ε in (3.7.10) can be restricted to $0<\varepsilon<1$. In fact, only small values of ε are important for this supremum. Accordingly, only values of r close to 1 matter in the two suprema in (3.7.1). This property can be also verified by the argument in the proof of Lemma 3.7.6 below.

Functions on $\mathbb{T}^d$ are 1-periodic functions on $\mathbb{R}^d$, or equivalently, functions on $\mathbb{I}^d$ can be extended to 1-periodic functions to $\mathbb{R}^d$. For a function f on $\mathbb{T}^d$ (or $\mathbb{I}^d$) $\tilde f$ will denote the corresponding 1-periodic function on $\mathbb{R}^d$. Then (3.7.8) implies that $\mathbb{Q}_\varepsilon[f]$ is equal to the Poisson integral of $\tilde f$ on $\mathbb{R}^d$ that will be denoted by $\varphi_\varepsilon[\tilde f]$. Let us record this useful fact here for later reference:

$$\mathbb{Q}_\varepsilon[f]=\varphi_\varepsilon[\tilde f]=\varphi_\varepsilon*\tilde f \quad \text{on} \quad \mathbb{I}^d. \tag{3.7.11}$$

Recall that $\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})$ is defined as the space of all locally square integrable functions ψ from $\mathbb{R}^d$ to $L_2(\mathcal{M})$ such that

$$\|\psi\|_{\mathrm{BMO}^c}=\max\Big\{\|\psi_{\mathbb{I}^d}\|_\infty,\quad \sup_{Q\subset\mathbb{R}^d\text{cube}}\Big\|\frac{1}{|Q|}\int_Q|\psi-\psi_Q|^2ds\Big\|_\infty^{1/2}\Big\}.$$

The following lemma shows that the map $f\mapsto\tilde f$ establishes an isomorphic embedding of $\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})$ into $\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})$.

Lemma 3.7.5. *For any $f\in\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})$ we have*

$$\|f\|_{\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})}\approx\|\tilde f\|_{\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})}$$

with equivalence constants depending only on d.

Proof. Let $f\in L_2(\mathbb{T}^d;L_2(\mathcal{M}))$ with $f_{\mathbb{I}^d}=0$. By (3.7.10) and (3.7.11), we have

$$\|f\|^2_{\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})}\approx\sup_{\varepsilon>0}\sup_{s\in\mathbb{R}^d}\|\varphi_\varepsilon[|\tilde f-\varphi_\varepsilon[\tilde f](s)|^2](s)\|_\infty.$$

Then the proof of Lemma 3.7.3 shows that the right hand side above is equivalent to $\|\tilde f\|^2_{\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})}$. Alternately, one can directly prove that the supremum on the right hand side in (3.7.10) is equivalent to $\|\tilde f\|^2_{\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})}$. Namely,

$$\sup_{Q\subset\mathbb{I}^d\text{cube}}\Big\|\frac{1}{|Q|}\int_Q|\tilde f-\tilde f_Q|^2ds\Big\|_\infty\approx\sup_{Q\subset\mathbb{R}^d\text{cube}}\Big\|\frac{1}{|Q|}\int_Q|\tilde f-\tilde f_Q|^2ds\Big\|_\infty.$$

Indeed, let Q be a cube in $\mathbb{R}^d$. If $|Q|\leq 1$, then by the definition of cubes in $\mathbb{I}^d$ and the periodicity of $\tilde f$, Q can be considered as a cube in $\mathbb{I}^d$. So assume $|Q|>1$. Take another cube R such that $Q\subset R$, $|R|\leq 2^d|Q|$ and the side length of R is an integer k. Then R is a union of k^d cubes of side length 1. Thus by the periodicity of $\tilde f$

$$\frac{1}{|Q|}\int_Q|\tilde f-\tilde f_Q|^2ds\leq 4\frac{1}{|Q|}\int_Q|\tilde f|^2ds\leq\frac{2^{d+2}}{|R|}\int_R|\tilde f|^2ds=2^{d+2}\int_{\mathbb{I}^d}|\tilde f|^2ds.$$

Therefore, we get the desired equivalence. □

Now we turn to the discussion of Hardy spaces. Let $1 \le p < \infty$. For $f \in L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$ define

$$G_c(f)(z) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[f](z)\Big|^2 (1-r)dr\Big)^{1/2}, \quad z \in \mathbb{T}^d \tag{3.7.12}$$

and

$$\|f\|_{\mathrm{H}_p^c} = \|\hat{f}(\mathbf{0})\|_p + \|G_c(f)\|_p.$$

Here the first L_p-norm is the one of $L_p(\mathcal{M})$ and the second that of $L_p(\mathbb{T}^d; L_p(\mathcal{M}))$. Completing $L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$ under the norm $\|\ \|_{\mathrm{H}_p^c}$, we get $\mathrm{H}_p^c(\mathbb{T}^d;\mathcal{M})$. Like in the BMO case, we wish to reduce these Hardy spaces to those on $\mathbb{I}^d$. Using the kernel $\mathbb{Q}_\varepsilon$ in (3.7.8), for $f \in L_\infty(\mathbb{I}^d)\overline{\otimes}\mathcal{M}$ let

$$\widetilde{G}_c(f)(s) = \Big(\int_0^\infty \Big|\frac{d}{d\varepsilon}\mathbb{Q}_\varepsilon[f](s)\Big|^2 \varepsilon d\varepsilon\Big)^{1/2}, \quad s \in \mathbb{I}^d. \tag{3.7.13}$$

Let $\tilde{f}$ be the periodic extension of f to $\mathbb{R}^d$. Let $\widetilde{G}_c(\tilde{f})$ be the g-function of $\tilde{f}$ defined by the Poisson kernel φ_ε:

$$\widetilde{G}_c(\tilde{f})(s) = \Big(\int_0^\infty \Big|\frac{d}{d\varepsilon}\varphi_\varepsilon[\tilde{f}](s)\Big|^2 \varepsilon d\varepsilon\Big)^{1/2}, \quad s \in \mathbb{R}^d. \tag{3.7.14}$$

Thanks to (3.7.11), we have

$$\widetilde{G}_c(f) = \widetilde{G}_c(\tilde{f}) \quad \text{on } \mathbb{I}^d. \tag{3.7.15}$$

Thus $\widetilde{G}_c(\tilde{f})$ is the periodic extension to $\mathbb{R}^d$ of $\widetilde{G}_c(f)$. Let

$$\|f\|_{\mathrm{H}_p^c} = \|f_{\mathbb{I}^d}\|_p + \|\widetilde{G}_c(f)\|_p.$$

Here the first L_p-norm is the one of $L_p(\mathcal{M})$ and the second that of $L_p(\mathbb{I}^d; L_p(\mathcal{M}))$. Define $\mathrm{H}_p^c(\mathbb{I}^d;\mathcal{M})$ to be the completion of $(L_\infty(\mathbb{I}^d)\overline{\otimes}\mathcal{M}, \|\ \|_{\mathrm{H}_p^c})$.

Lemma 3.7.6. *Let $1 \le p < \infty$. Then $\mathrm{H}_p^c(\mathbb{T}^d;\mathcal{M}) = \mathrm{H}_p^c(\mathbb{I}^d;\mathcal{M})$ with equivalent norms.*

Proof. We first show that in the definition of the Littlewood-Paley function $G_c(f)$ in (3.7.12) only values of r close to 1 matter. More precisely, for any $0 < r_0 < 1$ setting

$$G_{c,r_0}(f)(z) = \Big(\int_{r_0}^1 \Big|\frac{d}{dr}\mathbb{P}_r[f](z)\Big|^2 (1-r)dr\Big)^{1/2},$$

we have

$$\|G_c(f)\|_p \approx \|G_{c,r_0}(f)\|_p,$$

where the equivalence constants depend only on d and r_0. To this end take any $0 \le r_0 < r_1 < 1$ and let

$$G_c'(f)(z) = \Big(\int_{r_0}^{r_1} \Big|\frac{d}{dr}\mathbb{P}_r[f](z)\Big|^2 (1-r)dr\Big)^{1/2}.$$

We claim that

$$\|G_c'(f)\|_p \approx \sup_{n\in\mathbb{Z}^d, n\neq\mathbf{0}} \|\hat{f}(n)\|_p. \tag{3.7.16}$$

Writing the Fourier series expansion of $\mathbb{P}_r[f]$:

$$\mathbb{P}_r[f](z) = \sum_{n\in\mathbb{Z}^d} r^{|n|_2}\hat{f}(n)z^n,$$

we have

$$\frac{d}{dr}\mathbb{P}_r[f](z) = \sum_{n\in\mathbb{Z}^d, n\neq \mathbf{0}} |n|_2 r^{|n|_2-1}\hat{f}(n)z^n.$$

We then easily get the upper estimate of (3.7.16). To show the lower one, for $n \in \mathbb{Z}^d$, $n \neq \mathbf{0}$ we have

$$|n|_2 r^{|n|_2-1}\hat{f}(n)z^n = \int_{\mathbb{T}^d} \frac{d}{dr}\mathbb{P}_r[f](z\cdot w)w^{-n}dm(w).$$

Let $H = L_2((r_0, r_1); (1-r)dr)$. It is clear that for any $z \in \mathbb{T}^d$ we have

$$\Big(\int_{r_0}^{r_1} \big||n|_2 r^{|n|_2-1}\hat{f}(n)z^n\big|^2(1-r)dr\Big)^{1/2} \approx |\hat{f}(n)|.$$

Then by the triangle inequality in the column L_p-space $L_p(L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}; H^c)$, we deduce

$$\|\hat{f}(n)\|_{L_p(\mathcal{M})} \lesssim \int_{\mathbb{T}^d}\Big(\int_{\mathbb{T}^d}\|G_c'(f)(z\cdot w)\|^p_{L_p(\mathcal{M})}dm(z)\Big)^{1/p}dm(w) = \|G_c'(f)\|_{L_p(\mathbb{T}^d;L_p(\mathcal{M}))}.$$

Thus the claim is proved. Using (3.7.16) twice, we get

$$\begin{aligned}\|G_c(f)\|_p &\leq \Big\|\Big(\int_0^{r_0}\Big|\frac{d}{dr}\mathbb{P}_r[f]\Big|^2(1-r)dr\Big)^{1/2}\Big\|_p + \|G_{c,r_0}(f)\|_p\\ &\lesssim \sup_{n\in\mathbb{Z}^d, n\neq\mathbf{0}}\|\hat{f}(n)\|_p + \|G_{c,r_0}(f)\|_p \lesssim \|G_{c,r_0}(f)\|_p.\end{aligned}$$

Similarly, we show that only small values of ε matter in (3.7.13) and (3.7.14). Namely for $0 < \varepsilon_0 < \infty$ letting

$$\widetilde{G}_{c,\varepsilon_0}(f)(s) = \Big(\int_0^{\varepsilon_0}\Big|\frac{d}{d\varepsilon}\mathbb{Q}_\varepsilon[f](s)\Big|^2\varepsilon d\varepsilon\Big)^{1/2}, \quad s\in\mathbb{I}^d,$$

we have

$$\|\widetilde{G}_c(f)\|_p \approx \|\widetilde{G}_{c,\varepsilon_0}(f)\|_p.$$

Now it is easy to finish the proof of the lemma. Indeed, using the change of variables $r = e^{-2\pi\varepsilon}$, we get

$$\begin{aligned}G_{c,r_0}(f)(z) &= \frac{1}{2\pi}\Big(\int_0^{\varepsilon_0}\Big|\frac{d}{d\varepsilon}\mathbb{Q}_\varepsilon[f](s)\Big|^2 e^{2\pi\varepsilon}(1-e^{-2\pi\varepsilon})d\varepsilon\Big)^{1/2}\\ &\approx \Big(\int_0^{\varepsilon_0}\Big|\frac{d}{d\varepsilon}\mathbb{Q}_\varepsilon[f](s)\Big|^2\varepsilon d\varepsilon\Big)^{1/2} = \widetilde{G}_{c,\varepsilon_0}[f](s).\end{aligned}$$

Together with the previous equivalences, this implies the desired assertion. □

We will also need the Lusin area integral function. For $\alpha > 1$ and $z \in \mathbb{T}^d$, let $\mathbb{D}_\alpha(z)$ be the Stoltz domain with vertex z and aperture α (recalling that $|w|$ denotes the Euclidean norm):

$$\mathbb{D}_\alpha(z) = \{w \in \mathbb{C}^d \ : \ |z-w| \leq \alpha(1-|w|)\}.$$

For $f \in L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$ define

$$S_c^\alpha(f)(z) = \Big(\int_{\mathbb{D}_\alpha(z)}\Big|\frac{d}{dr}\mathbb{P}_r[f](rw)\Big|^2\frac{dm(w)dr}{(1-r)^{d-1}}\Big)^{1/2}, \quad z\in\mathbb{T}^d, \tag{3.7.17}$$

where the integral is taken over $\mathbb{D}_\alpha(z)$ with respect to $rw \in \mathbb{D}_\alpha(z)$ with $0 \leq r < 1$ and $w \in \mathbb{T}^d$ (recalling that dm is Haar measure of $\mathbb{T}^d$).

Like for the g-function, we will transfer $S_c^\alpha(f)$ to the usual area integral function on $\mathbb{R}^d$. For $\beta > 0$ and $s \in \mathbb{R}^d$ let

$$\Gamma_\beta(s) = \{(t,\varepsilon) \in \mathbb{R}^d \times \mathbb{R}_+ \ : \ |t-s| \le \beta\varepsilon\}.$$

Let $f \in L_\infty(\mathbb{I}^d)\overline{\otimes}\mathcal{M}$ and $\tilde{f}$ be its periodic extension to $\mathbb{R}^d$. Define

$$\begin{aligned} \widetilde{S}_c^\beta(f)(s) &= \Big(\int_{\Gamma_\beta(s)} \Big|\frac{d}{d\varepsilon}\mathbb{Q}_\varepsilon[f](t)\Big|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\Big)^{1/2} \\ &= \Big(\int_{\Gamma_\beta(s)} \Big|\frac{d}{d\varepsilon}\varphi_\varepsilon[\tilde{f}](t)\Big|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\Big)^{1/2} = \widetilde{S}_c^\beta(\tilde{f})(s). \end{aligned} \tag{3.7.18}$$

The following is the analogue of Lemma 3.7.6 for the Lusin square functions.

Lemma 3.7.7. *Let $\alpha > 1$ and $\beta > 0$. Let $f \in L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$. Then*

$$\|S_c^\alpha(f)\|_{L_p(\mathbb{T}^d;L_p(\mathcal{M}))} \approx \|\widetilde{S}_c^\beta(f)\|_{L_p(\mathbb{I}^d;L_p(\mathcal{M}))}$$

with equivalence constants depending only on d and α, β. Moreover, the norms above are independent of α and β up to equivalence.

Proof. This proof is similar to that of Lemma 3.7.6. For $0 < r_0 < 1$ we introduce the truncated Stoltz domain:

$$\mathbb{D}_{\alpha,r_0}(z) = \{w \in \mathbb{C}^d \ : \ |z-w| \le \alpha(1-|w|),\, r_0 < |w| < 1\}.$$

Also for $\varepsilon_0 > 0$ set

$$\Gamma_{\beta,\varepsilon_0}(s) = \{(t,\varepsilon) \in \mathbb{R}^d \times \mathbb{R}_+ \ : \ |t-s| \le \beta\varepsilon,\, \varepsilon < \varepsilon_0\}.$$

The corresponding truncated square functions are

$$S_{c,r_0}^\alpha(f)(z) = \Big(\int_{\mathbb{D}_{\alpha,r_0}(z)} \Big|\frac{d}{dr}\mathbb{P}_r[f](rw)\Big|^2 \frac{dm(w)dr}{(1-r)^{d-1}}\Big)^{1/2}$$

and

$$\widetilde{S}_{c,\varepsilon_0}^\beta(f)(s) = \Big(\int_{\Gamma_{\beta,\varepsilon_0}(s)} \Big|\frac{d}{d\varepsilon}\varphi_\varepsilon[\tilde{f}](t)\Big|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\Big)^{1/2}.$$

Then by the reasoning in the proof of Lemma 3.7.6, we have

$$\|S_c^\alpha(f)\|_p \approx \|S_{c,r_0}^\alpha(f)\|_p$$

and a similar equivalence for $\widetilde{S}_c^\beta(f)$. On the other hand, it is easy to see that for any $\alpha > 1$ and $0 < r_0 < 1$ there exist $\beta_1, \beta_2 > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that under the change of variables $r = e^{-2\pi\varepsilon}$ and $w = e^{-2\pi i t}$

$$\Gamma_{\beta_1,\varepsilon_1}(s) \subset \mathbb{D}_{\alpha,r_0}(z) \subset \Gamma_{\beta_2,\varepsilon_2}(s), \quad \forall\, z = e^{-2\pi i s} \in \mathbb{T}^d.$$

Conversely, every truncated cone $\Gamma_{\beta,\varepsilon_0}(s)$ is located between two truncated Stoltz domains. Then the argument at the end of the proof of Lemma 3.7.6 implies

$$\widetilde{S}_{c,\varepsilon_1}^{\beta_1}(f)(s) \lesssim S_{c,r_0}^\alpha(f)(z) \lesssim \widetilde{S}_{c,\varepsilon_2}^{\beta_2}(f)(s);$$

whence

$$\|\widetilde{S}^{\beta_1}_{c,\varepsilon_1}(f)\|_{L_p(\mathbb{I}^d;L_p(\mathcal{M}))} \lesssim \|S^{\alpha}_{c,r_0}(f)\|_{L_p(\mathbb{T}^d;L_p(\mathcal{M}))} \lesssim \|\widetilde{S}^{\beta_2}_{c,\varepsilon_2}(f)\|_{L_p(\mathbb{I}^d;L_p(\mathcal{M}))}.$$

However, standard arguments in harmonic analysis show that

$$\|\widetilde{S}^{\beta_1}_{c}(f)\|_{L_p(\mathbb{I}^d;L_p(\mathcal{M}))} \approx \|\widetilde{S}^{\beta_2}_{c}(f)\|_{L_p(\mathbb{I}^d;L_p(\mathcal{M}))},$$

where the equivalence constants depend on d and β_1, β_2 (cf. e.g., [13]). Therefore, we deduce the first equivalence assertion of the lemma. The second part then follows too. □

Now we can show that the results of [52] remain valid for $\mathbb{T}^d$ too. We state only those relevant to Theorem 3.7.1. In the following statement, the row and mixture Hardy/BMO spaces are defined in the usual way, and $S_c(f) = S^2_c(f)$, $\widetilde{S}_c(f) = \widetilde{S}^1_c(f)$.

Theorem 3.7.8. i) *The dual space of* $\mathrm{H}^c_1(\mathbb{T}^d;\mathcal{M})$ *coincides with* $\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M})$ *isomorphically with the natural duality bracket. The same assertion holds for the row and mixture spaces.*

ii) *Let* $1 \le p < \infty$. *Then for any* $f \in L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$

$$\big\|G_c(f)\big\|_p \approx \big\|S_c(f)\big\|_p$$

with relevant constants depending only on d. Consequently, the two square functions G_c *and* S_c *define a same Hardy space.*

iii) *Let* $1 < p < \infty$. *Then* $\mathrm{H}_p(\mathbb{T}^d;\mathcal{M}) = L_p(\mathbb{T}^d;L_p(\mathcal{M}))$ *with equivalent norms.*

iv) *Let* $1 < p < \infty$. *Then*

$$(\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M}),\ \mathrm{H}^c_1(\mathbb{T}^d;\mathcal{M}))_{1/p} = \mathrm{H}^c_p(\mathbb{T}^d;\mathcal{M}) = (\mathrm{BMO}^c(\mathbb{T}^d;\mathcal{M}),\ \mathrm{H}^c_1(\mathbb{T}^d;\mathcal{M}))_{1/p,p}\,.$$

v) *Let* $X_0 \in \{\mathrm{BMO}(\mathbb{T}^d;\mathcal{M}),\ L_\infty(\mathbb{T}^d;L_p(\mathcal{M}))\}$, $X_1 \in \{\mathrm{H}_1(\mathbb{T}^d;\mathcal{M}),\ L_1(\mathbb{T}^d;\mathcal{M})\}$. *Then for any* $1 < p < \infty$

$$(X_0,\ X_1)_{1/p} = L_p(\mathbb{T}^d;\mathcal{M}) = (X_0,\ X_1)_{1/p,p}\,.$$

Proof. By the identification $\mathbb{T}^d \cong \mathbb{I}^d$ and Lemmas 3.7.3, 3.7.6 and 3.7.7, it suffices to prove this theorem with $\mathbb{I}^d$ instead of $\mathbb{T}^d$. The geometry of $\mathbb{I}^d$ is closer to that of $\mathbb{R}^d$. However, what makes our arguments parallel to those of [52] is the use of periodic functions. This periodization puts the arguments of [52] directly at our disposal. For any function f on $\mathbb{I}^d$ with periodic extension $\tilde{f}$ to $\mathbb{R}^d$, by (3.7.15) and (3.7.18), we have

$$\widetilde{G}_c(f) = \widetilde{G}_c(\tilde{f}) \quad \text{and} \quad \widetilde{S}_c(f) = \widetilde{S}_c(\tilde{f}) \ \text{ on } \mathbb{I}^d.$$

Note that the two square functions on the right are exactly those introduced in [52] by using the Poisson kernel on $\mathbb{R}^d$. The only difference compared with [52] is that the L_p-norm of these square functions are now taken in $L_p(\mathbb{I}^d;L_p(\mathcal{M}))$ instead of $L_p(\mathbb{R}^d;L_p(\mathcal{M}))$. In other words, the integral is now taken on $\mathbb{I}^d$ instead of $\mathbb{R}^d$. On the other hand, by Lemmas 3.7.3 and 3.7.5, the map $f \mapsto \tilde{f}$ is an isomorphic embedding of $\mathrm{BMO}^c(\mathbb{I}^d;\mathcal{M})$ into $\mathrm{BMO}^c(\mathbb{R}^d;\mathcal{M})$. It is now easy to see that the proof of [52, Theorem 2.4] is valid for periodic functions and integration on $\mathbb{I}^d$. Hence, we get the duality result in part i) and the equivalence for $p = 1$ in part ii). In the same way, we prove the periodic analogue of [52, Theorem 4.4], which implies the remaining case of ii). The reduction to dyadic martingales of [52] is clearly available in our present setting. The dyadic decomposition is now made in $\mathbb{I}^d$ (or equivalently, $\mathbb{T}^d$). In this way, we reduce parts iii)-v) to the martingale case as in [52]. The verification of all details is, however, tedious and lengthy, so it is more reasonable to skip it here. $\square$

Remark 3.7.9. It is stated in the final remark of [52, Chapter 2] that the relevant constants in part i) above for $\mathbb{R}^d$ are independent of d. This does not seem true. In fact, all constants appearing in Theorem 3.7.8 depend on d (and on p too), except those in part iii) since the semigroup argument described in the paragraph following Theorem 3.7.1 yields equivalence constants depending only on p. The same comment applies to Theorem 3.7.1 too. However, the constants there are independent of the given skew matrix θ.

Remark 3.7.10. The H_1-BMO duality in Theorem 3.7.8 and Lemma 3.7.3 imply that $\mathrm{H}_1^c(\mathbb{T}^d;\mathcal{M})$ admits an atomic decomposition like in the case of $\mathbb{R}^d$. We refer the interested reader to [52] for more details.

Armed with Theorem 3.7.8 and transference, it is easy to prove Theorem 3.7.1. To this end we still require the following simple lemma.

Lemma 3.7.11. *The map $x \mapsto \tilde{x}$ in Corollary 3.1.2 extends to an isometric embedding from $\mathrm{H}_p^c(\mathbb{T}_\theta^d)$ into $\mathrm{H}_p^c(\mathbb{T}^d;\mathbb{T}_\theta^d)$ for any $1 \leq p < \infty$ and from $\mathrm{BMO}_c(\mathbb{T}_\theta^d)$ into $\mathrm{BMO}_c(\mathbb{T}^d;\mathbb{T}_\theta^d)$. Moreover, the images of this embedding are 1-complemented in their respective spaces.*

Proof. The first part follows immediately from the identity $\mathbb{P}_r[\tilde{x}] = \widetilde{\mathbb{P}_r[x]}$ for any $x \in L_1(\mathbb{T}_\theta^d)$. Identifying $\widetilde{\mathbb{T}_\theta^d}$ with $\mathbb{T}_\theta^d$, the conditional expectation $\mathbb{E}$ from $L_\infty(\mathbb{T}^d)\overline{\otimes}\mathbb{T}_\theta^d$ onto $\mathbb{T}_\theta^d$ extends to a contractive projection from $\mathrm{H}_p^c(\mathbb{T}^d;\mathbb{T}_\theta^d)$ onto $\mathrm{H}_p^c(\mathbb{T}_\theta^d)$ and from $\mathrm{BMO}_c(\mathbb{T}^d;\mathbb{T}_\theta^d)$ onto $\mathrm{BMO}_c(\mathbb{T}_\theta^d)$. This yields the second part. $\square$

The proof of Theorem 3.7.1. It is now clear that Theorem 3.7.1 follows immediately from Theorem 3.7.8 (with $\mathcal{M} = \mathbb{T}_\theta^d$) and Lemma 3.7.11. $\square$

Remark 3.7.12. Since $\mathbb{T}_\theta^d \subset \mathrm{BMO}(\mathbb{T}_\theta^d)$, part ii) of Theorem 3.7.1 implies that $\mathrm{H}_1(\mathbb{T}_\theta^d) \subset L_1(\mathbb{T}_\theta^d)$ and

$$\|x\|_1 \leq C\|x\|_{\mathrm{H}_1}, \quad \forall\, x \in \mathrm{H}_1(\mathbb{T}_\theta^d).$$

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